

# A CATEGORICAL RECONSTRUCTION OF CRYSTALS AND QUANTUM GROUPS AT $q = 0$

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**ABSTRACT.** Our goal is to endow some crystal bases with the structure of a bialgebra, with the hope of classifying crystals as comodules over a crystal bialgebra. Conceptually, we may think of these algebraic structures as quantum groups over the hypothetical field with one element. We then move to the theory of comonadic functors, giving a classification of crystal bases as coalgebras over a comonadic functor, which we then link back to the attempts from the first section. We also encode the monoidal structure of the category of crystals into our comonadic functor, giving a bi(co)monadic functor. In Part 3 we alter the situation and work with linear combinations of Crystal basis elements, which resolves some of the issues we encountered in the first part. We finish by suggesting a link between this work and Lusztig's Quantum Group at  $v = \infty$ .

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## 0. INTRODUCTION

The goal of parts 1 and 2 was to endow some crystal bases with the structure of a bialgebra. Conceptually, we may think of these algebraic structures as quantum groups over the hypothetical field with one element. In Part 3 we alter the situation and work with 'linearised crystals'. Working with linear combinations of Crystal basis elements resolves some of the issues we encountered in the first part.

In the first section, we study the algebraic structures that are induced on certain  $U_q(\mathfrak{sl}_2)$ -modules by the Clebsch-Gordan decomposition, specifically that of the space spanned by matrix coefficients of the irreducible  $\mathfrak{sl}_2$  representations. This space, which can also be viewed as the space of algebraic functions on the group  $SL(2; \mathbb{C})$ , forms a bialgebra whose comodules are precisely representations of  $\mathfrak{sl}_2$ . The main aim of this section is to see whether this structure remains when we move to crystal bases, and whether we obtain a similar classification result. We find that we retain some, but not all, of this structure and that crystal bases can not be classified in the same way. Much of this work is done for crystal bases over general Kac-Moody Lie algebras, although, in the case of  $\mathfrak{sl}_2$ , our results can be explicitly computed.

The second section has more of a categorical feel. After initially failing to classify crystal bases as comodules over an abstract algebra in the category of crystals, we turn to category theory and (co)monadic functors. This gives the classification of crystal bases as coalgebras over a comonadic functor, which we link back to the attempts from the first section. We finish by encoding the monoidal structure of the category of crystals into our comonadic functor, giving some form of bimonadic functor. This is done by applying an extension of the Barr-Beck Theorem to monadic functors on monoidal categories. All of this work is done for a crystal bases over an arbitrary

Kac–Moody Lie algebra.

In the third section we work with free abelian groups on crystal bases, which allows us to avoid some of the issues from Part 1. We then define a bialgebra whose comodules are precisely free abelian groups on crystal bases and study some of its properties. We provide a comparison between the functor from Part 2 and the functor obtained by tensoring with this bialgebra. After a brief discussion about the dual to this bialgebra, we then outline a way of perhaps relating the structure of this bialgebra to that of the quantum co-ordinate ring, and linking this work to a paper by Lusztig in which he constructs a similar object.

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## Part 1. Crystal Bialgebras

### 1. QUANTUM GROUPS AND CRYSTALS

**1.1. Quantum Groups.** We begin by setting some notation and recalling some preliminary results. The following well known constructions of general quantum groups can be seen in Kashiwara’s paper [8] and in Jantzen’s book [4, p. 51], or for a more detailed account, see Lusztig’s book [12].

**Definition** Suppose a Lie algebra  $\mathfrak{g}$  is defined by the following data:

- i) A weight lattice  $\Phi$ , a free  $\mathbb{Z}$ -module, with simple roots  $\alpha_i \in \Phi$  indexed by  $i$  in some indexing set  $I$  that form a basis of the root lattice  $\Psi$  (with respect to some Cartan subalgebra) contained in  $\Phi$ ;
- ii) A symmetric bilinear form  $(\cdot, \cdot) : \Phi \times \Phi \rightarrow \mathbb{Q}$  such that  $(\alpha_i, \alpha_i) \in 2\mathbb{N}$ ,  $(\alpha_i, \alpha_j) \leq 0$  for  $i, j \in I, i \neq j$ ;
- iii) Simple coroots  $\lambda_i \in \Phi^* = \text{Hom}_{\mathbb{Z}}(\Phi, \mathbb{Z})$  such that  $\lambda_i(\alpha) = \frac{2(\alpha_i, \alpha)}{(\alpha_i, \alpha_i)}$  for  $i \in I, \alpha \in \Phi$ .

Then  $\mathfrak{g}$  can be generated by  $e_i, f_i, h_i$  for  $i \in I$  with the relations

$$[h_i, h_j] = 0, \quad [e_i, f_i] = \delta_{ij} h_i, \quad [h_i, e_j] = \lambda_i(\alpha_j) e_j, \quad [h_i, f_j] = -\lambda_i(\alpha_j) f_j,$$

and for  $i \neq j$ ,

$$(\text{ad } e_i)^{1-\lambda_i(\alpha_j)} e_j = 0, \quad (\text{ad } f_i)^{1-\lambda_i(\alpha_j)} f_j = 0,$$

where  $\text{ad}$  is the adjoint map  $(\text{ad } x)(y) = [x, y]$ .

**Definition** We will denote by  $\Psi_+ = \{\sum_{i \in I} n_i \alpha_i \mid n_i \geq 0\} \subset \Psi$  the positive roots, and  $\Psi_- = -\Psi_+$  the negative roots. Let  $\Phi_+ = \{\alpha \in \Phi \mid \lambda_i(\alpha) \geq 0 \text{ for all } i \in I\}$  be the dominant weights, and likewise  $\Phi_-$  the anti-dominant weights. Then  $\Phi$  has a partial ordering given by  $\alpha \geq \beta$  if and only if  $\alpha - \beta \in \Phi_+$ .

**Remark** For example in the case of  $\mathfrak{sl}_2$  the weight lattice  $\Phi$  is  $\mathbb{Z}$  with a single simple root 2.

**Definition** Let  $W$  denote the *Weyl group* attached to this data. That is,  $W$  is the subgroup of  $GL(\Phi)$  generated by the *simple reflections*  $s_i(\alpha) = \alpha - \frac{2(\alpha_i, \alpha)}{(\alpha_i, \alpha_i)} \alpha_i = \alpha - \lambda_i(\alpha) \alpha_i$  for  $i \in I$ . This forms a coxeter group, and so has a unique element of highest length when written in terms of these simple reflections, which we shall denote as  $w_0$ .

**Definition** Take  $q$  to be a general nonzero element of our base field  $k$  which is not a root of unity. It will be convenient to think of  $q$  as an independent indeterminant and work over  $K = k(q)$ . We may then define the *quantised enveloping algebra*  $U_q(\mathfrak{g})$  to be the algebra generated over our field  $k$  by  $e_i, f_i, q^\lambda$  for  $i \in I, \lambda \in \Phi^*$ , with the defining relations

$$\begin{aligned} & \text{for } \lambda = 0 \quad q^\lambda = 1, \\ & \text{for } \lambda_1, \lambda_2 \in \Phi^* \quad q^{\lambda_1} q^{\lambda_2} = q^{\lambda_1 + \lambda_2}, \\ & \text{for } i \in I, \lambda \in \Phi^* \quad q^\lambda e_i q^{-\lambda} = q^{\lambda(\alpha_i)} e_i, \\ & \quad q^\lambda f_i q^{-\lambda} = q^{-\lambda(\alpha_i)} f_i, \\ & \quad e_i f_i - f_i e_i = \delta_{ij} \frac{t_i - t_i^{-1}}{q_i - q_i^{-1}}, \\ & \text{for } i \neq j \quad \sum_{k=0}^{1-\lambda_i(\alpha_j)} (-1)^k e_i^{(k)} e_j e_i^{(1-\lambda_i(\alpha_j)-k)} \\ & \quad = \sum_{k=0}^{1-\lambda_i(\alpha_j)} (-1)^k f_i^{(k)} f_j f_i^{(1-\lambda_i(\alpha_j)-k)} = 0 \end{aligned}$$

where  $q_i = q^{\frac{(\alpha_i, \alpha_i)}{2}}$ ,  $t_i = q^{\frac{(\alpha_i, \alpha_i)}{2} \lambda_i}$ ,  $[k]_i = \frac{q_i^k - q_i^{-k}}{q_i - q_i^{-1}}$ ,  $[k]_i! = [1]_i [2]_i \dots [k]_i$ ,

$f_i^{(k)} = f_i^k / [k]_i!$ , and  $e_i^{(k)} = e_i^k / [k]_i!$ . The last set of relations above are known as the *quantum Serre relations*. Let us denote by  $U_q(\mathfrak{n})$  (respectively  $U_q(\mathfrak{n}^-)$ ) the subalgebra of  $U_q(\mathfrak{g})$  generated by  $\{e_i \mid i \in I\}$  (respectively  $\{f_i \mid i \in I\}$ ). Similarly let  $U_q(\mathfrak{b})$  (respectively  $U_q(\mathfrak{b}^-)$ ) be the subalgebra of  $U_q(\mathfrak{g})$  generated by  $\{e_i \mid i \in I\} \cup \{q^\lambda \mid \lambda \in \Phi^*\}$  (respectively  $\{f_i \mid i \in I\} \cup \{q^\lambda \mid \lambda \in \Phi^*\}$ ).

**Remark** Returning to our example of  $\mathfrak{sl}_2$ ,  $U_q(\mathfrak{sl}_2)$  is the  $k$ -algebra generated by  $e, f, t, t^{-1}$  with defining relations

$$tet^{-1} = q^2 e, \quad tft^{-1} = q^{-2} f, \quad ef - fe = \frac{t - t^{-1}}{q - q^{-1}}.$$

We can see that the subalgebras of  $U_q(\mathfrak{g})$  generated by  $e_i, f_i, t_i, t_i^{-1}$ , denoted  $U_q(\mathfrak{g})_i$ , are isomorphic to  $U_q(\mathfrak{sl}_2)$ . So we may build these quantum groups up from quantised  $\mathfrak{sl}_2$ , which we will often refer to as our main example.

**Definition** We say that a left  $U_q(\mathfrak{g})$  module  $M$  is *integrable* if  $M$  decomposes into *weight spaces*  $M = \bigoplus_{\alpha \in \Phi} M_\alpha$ ,  $M_\alpha = \{m \in M \mid q^\lambda m = q^{\lambda(\alpha)} m \text{ for all } \lambda \in \Phi^*\}$ , and for each  $i \in I$   $M$  is a locally finite dimensional  $U_q(\mathfrak{g})_i$  module. We

then define  $\mathcal{O}_{\mathfrak{g}}$  to be the category of integrable left  $U_q(\mathfrak{g})$  modules that are locally finite dimensional as  $U_q(\mathfrak{n})$  modules. Likewise we define integrable right  $U_q(\mathfrak{g})$  modules, and an analogous category  $\mathcal{O}_{\mathfrak{g}^{\text{op}}}$ .

**Proposition 1.1** (Lusztig, [12]). *Objects in  $\mathcal{O}_{\mathfrak{g}}$  are completely reducible, and all irreducible objects are, up to isomorphism, indexed by  $\alpha \in \Phi^+$ . These irreducible, usually denoted  $V(\alpha)$ , can be expressed explicitly as the representation generated by a single vector  $u_\alpha$ , called the highest weight vector, with the defining relations*

$$e_i u_\alpha = 0 = f_i^{1+\lambda_i(\alpha)} u_\alpha, q^\lambda u_\alpha = q^{\lambda(\alpha)} u_\alpha, \quad i \in I, \lambda \in \Phi.$$

**Remark** It is well known that the dual representation to  $V(\alpha)$ ,  $V(\alpha)^\vee$ , is isomorphic to the irreducible representation  $V(w_0\alpha)$ .

**Remark** In the case of  $\mathfrak{sl}_2$ , these irreducible representations are  $V(n)$  indexed by  $n \in \mathbb{Z}_{\geq 0}$ . They have a basis  $B(n) = \{u_i^{(n)} \mid 0 \leq i \leq n\}$  of  $t$ -eigenvectors with

$$tu_i^{(n)} = q^{n-2i} u_i^{(n)}, eu_i^{(n)} = [n-i+1] u_{i-1}^{(n)}, fu_i^{(n)} = [i+1] u_{i+1}^{(n)}.$$

Here we use the notation  $[n] = \frac{q^n - q^{-n}}{q - q^{-1}}$ . So, up to scalar multiplication,  $e$  decreases the index  $i$ , whilst  $f$  increases the index. Thus we may define operators  $\tilde{e} : u_i^{(n)} \mapsto u_{i-1}^{(n)}$  and  $\tilde{f} : u_i^{(n)} \mapsto u_{i+1}^{(n)}$  on the basis  $B(n)$ . These are often referred to as the *Kashiwara operators* and important to the study of crystal bases.

**Definition** Let  $A_q(\mathfrak{g})$  denote the *quantum co-ordinate algebra* defined by the direct sum  $A_q(\mathfrak{g}) = \bigoplus_{\alpha \in \Phi^+} V(\alpha) \otimes V(\alpha)^\vee$  with unit  $1 = v_0 \otimes v_0 \in V(0) \otimes V(0)^\vee$  and multiplication defined by the composition

$$\begin{aligned} V(\alpha) \otimes V(\alpha)^\vee \otimes V(\beta) \otimes V(\beta)^\vee &\rightarrow V(\alpha) \otimes V(\beta) \otimes V(\beta)^\vee \otimes V(\alpha)^\vee \\ &\rightarrow V(\alpha) \otimes V(\beta) \otimes (V(\alpha) \otimes V(\beta))^\vee \\ &\rightarrow \left( \bigoplus_{\gamma} V(\gamma) \right) \otimes \left( \bigoplus_{\delta} V(\delta) \right)^\vee \\ &\twoheadrightarrow \bigoplus_{\gamma} V(\gamma) \otimes V(\gamma)^\vee \end{aligned}$$

where the first arrow is given by exchanging the order of terms in the underlying tensor product of vector spaces (ie  $t \otimes u \otimes v \otimes w \mapsto t \otimes v \otimes w \otimes u$  on elementary tensors), and the third arrow is given by the decomposition into irreducible components. This algebra has a comultiplication given by

$$\begin{aligned} V(\alpha) \otimes V(\alpha)^\vee &\cong V(\alpha) \otimes k \otimes V(\alpha)^\vee \\ &\rightarrow V(\alpha) \otimes (V(\alpha)^\vee \otimes V(\alpha)) \otimes V(\alpha)^\vee \hookrightarrow A_q(\mathfrak{g}) \otimes A_q(\mathfrak{g}) \end{aligned}$$

induced by the coevaluation map, and counit given by the evaluation map.

**Remark** By the quantum Peter-Weyl Theorem (see, for example, [3]), this can be identified with a sub-bialgebra of functions on the quantum enveloping algebra  $U_q(\mathfrak{g})$ , where  $u \otimes v \in V(\alpha) \otimes V(\alpha)^\vee$  is seen as the function  $x \mapsto \langle x \cdot u, v \rangle$ . The image of  $A_q(\mathfrak{g})$  is then the sub-bialgebra of all functions in  $U_q(\mathfrak{g})^\vee$  such that the left and right  $U_q(\mathfrak{g})$  submodules of  $U_q(\mathfrak{g})^\vee$  they generate are in  $\mathcal{O}_{\mathfrak{g}}$  and  $\mathcal{O}_{\mathfrak{g}^{\text{op}}}$  respectively.

**1.2. The Category of Crystals.** We begin by describing the category of crystals, a generalisation of crystal bases, as Kashiwara defines in [8]. See *loc. cit.* for the motivation and intuition behind the following definitions.

**Definition** A *pointed set* is a set with a distinct point or element, which we shall denote 0,  $A_\bullet = A \sqcup \{0\}$ , with unions  $A_\bullet \cup B_\bullet = (A \sqcup \{0\}) \cup (B \sqcup \{0\}) := (A \cup B) \sqcup \{0\}$  and products  $A_\bullet \times B_\bullet = (A \times B) \sqcup \{0\}$ . A morphism between pointed sets  $A_\bullet$  and  $B_\bullet$  is a map from  $A$  to  $B \sqcup \{0\}$  extended to map  $0 \mapsto 0$ . This defines a category of pointed sets, which we shall denote  $\text{Set}_\bullet$ .

**Definition** We define the objects of the category of crystals, denoted  $\text{Crys}$ , to be pointed sets  $B$  equipped with maps

$$\begin{aligned} \tilde{e}_i &: B \rightarrow B, & \tilde{f}_i &: B \rightarrow B, \\ \varepsilon_i &: B \rightarrow \mathbb{Z} \sqcup \{-\infty\}, & \phi_i &: B \rightarrow \mathbb{Z} \sqcup \{-\infty\}, \\ \text{wt} &: B \rightarrow \Phi, \end{aligned}$$

for all  $i \in I$  such that, for a crystal  $B$  and  $b, b_1, b_2 \in B$ ,

$$\begin{aligned} & \phi_i(b) = \lambda_i(\text{wt}(b)) + \varepsilon_i(b), \\ \text{if } \tilde{e}_i(b) \neq 0 \text{ then } & \begin{aligned} \varepsilon_i(\tilde{e}_i b) &= \varepsilon_i(b) - 1, \\ \phi_i(\tilde{e}_i b) &= \phi_i(b) + 1, \\ \text{wt}(\tilde{e}_i b) &= \text{wt}(b) + \alpha_i, \end{aligned} \\ \text{if } \tilde{f}_i(b) \neq 0 \text{ then } & \begin{aligned} \varepsilon_i(\tilde{f}_i b) &= \varepsilon_i(b) + 1, \\ \phi_i(\tilde{f}_i b) &= \phi_i(b) - 1, \\ \text{wt}(\tilde{f}_i b) &= \text{wt}(b) - \alpha_i, \\ b_2 = \tilde{f}_i b_1 &\Leftrightarrow b_1 = \tilde{e}_i b_2, \end{aligned} \\ \text{if } \phi(b) = -\infty \text{ then } & \tilde{e}_i b = \tilde{f}_i b = 0, \end{aligned}$$

again, with the assumption that  $-\infty + n = -\infty$  for any  $n \in \mathbb{Z}$ . For crystals  $B_1, B_2$ , we say that a map  $\psi : B_1 \rightarrow B_2$  is a *morphism of crystals* if, for  $b \in B_1$ ,

$$\begin{aligned} & \psi(0) = 0, \\ \text{if } \psi(b) \neq 0 \text{ then } & \begin{aligned} \varepsilon_i(\psi(b)) &= \varepsilon_i(b), \\ \phi_i(\psi(b)) &= \phi_i(b), \\ \text{wt}(\psi(b)) &= \text{wt}(b), \end{aligned} \\ \text{if } \psi(b) \neq 0 \text{ and } \psi(\tilde{e}_i b) \neq 0 \text{ then } & \psi(\tilde{e}_i b) = \tilde{e}_i \psi(b), \\ \text{and if } \psi(b) \neq 0 \text{ and } \psi(\tilde{f}_i b) \neq 0 \text{ then } & \psi(\tilde{f}_i b) = \tilde{f}_i \psi(b). \end{aligned}$$

**Definition** We will call a crystal *finite* if its underlying pointed set is of finite cardinality. We say that a crystal is *irreducible* if it has no nontrivial proper subcrystals.

**Remark** Objects in this category have some nice combinatorial properties, as well as a graph structure. From each crystal  $B$  we obtain a *crystal graph* whose vertices are the nonzero points in  $B$  with arrows labeled by  $i \in I$ ,  $b \xrightarrow{i} b'$  if and only if  $b' = \tilde{f}_i b$ . Crystal graphs are made up of disjoint unions of connected components. It is clear that a crystal base is *irreducible* if and only if its crystal graph is connected.

**Remark** The main source of examples of objects in this category are crystal bases of integrable  $U_q(\mathfrak{g})$ -modules. We omit the rather involved definition of a crystal base and instead refer interested readers to [8]. What is important to note is that part of the data of a crystal base of a  $U_q(\mathfrak{g})$ -modules  $M$  is a pointed subset,  $B \subset M$ . The subset  $B$  is closed under the action of *Kashiwara operators*  $\tilde{e}_i$  and  $\tilde{f}_i$  for all  $i \in I$ , and each element  $b \in B$  is homogeneous with respect to the weight space decomposition, hence has an associated weight,  $\text{wt}(b)$ . Thus a crystal bases gives a crystal,  $B$ , in  $\text{Crys}$  where  $\varepsilon_i(b) = \max\{n \geq 0 \mid \tilde{e}_i^n(b) \neq 0\}$  and  $\phi_i(b) = \max\{n \geq 0 \mid \tilde{f}_i^n(b) \neq 0\}$  for  $b \in B(\alpha)$ .

**Theorem 1.2.** (Kashiwara [9]) *Each  $V(\alpha)$  has a unique crystal base, up to equivalence, with associated crystal  $B(\alpha)$  such that  $B(\alpha) \cap V(\alpha)_\alpha = \{u_\alpha\}$  and*

$$B(\alpha) = \{\tilde{f}_{i_1}^{n_1} \tilde{f}_{i_2}^{n_2} \dots \tilde{f}_{i_k}^{n_k} u_\alpha \mid i_1, i_2, \dots, i_k \in I, n_1, n_2, \dots, n_k \geq 0\}.$$

**Remark** Since  $U_q(\mathfrak{g})$  is semisimple, we see that any integrable  $U_q(\mathfrak{g})$ -module in  $\mathcal{O}_{\mathfrak{g}}$  has a unique crystal in  $\text{Crys}$  arising as a disjoint union of these  $B(\alpha)$ . We shall call such crystals the *crystals arising from integrable  $U_q(\mathfrak{g})$ -modules*.

**Remark** In the case where  $\mathfrak{g} = \mathfrak{sl}_2$ , each irreducible  $U_q(\mathfrak{sl}_2)$ -module  $V(n)$  has a corresponding crystal base  $\{u_k^{(n)}\}_{0 \leq k \leq n}$  which corresponds to a unique crystal, which we shall denote by  $B(n) = \{x^i y^{n-i} \mid 0 \leq i \leq n\}$  (here we identify  $u_i^{(n)}$  with  $x^i y^{n-i}$ ). This has crystal structure defined by

$$\begin{aligned} \tilde{f}(x^i y^{n-i}) &= x^{i+1} y^{n-i-1}, \quad \tilde{e}(x^i y^{n-i}) = x^{i-1} y^{n-i+1} \\ \varepsilon(x^i y^{n-i}) &= i, \quad \phi(x^i y^{n-i}) = n - i, \quad \text{wt}(x^i y^{n-i}) = n - 2i. \end{aligned}$$

So the crystal base of an irreducible  $U_q(\mathfrak{sl}_2)$ -module would have crystal graph

$$\begin{array}{c} \varepsilon(b) \qquad \qquad \psi(b) \\ \circ \xrightarrow{\quad} \circ \xrightarrow{\quad} \circ \dots \circ \xrightarrow{\quad} \circ \xrightarrow{\quad} b \xrightarrow{\quad} \circ \xrightarrow{\quad} \circ \dots \circ \xrightarrow{\quad} \circ \xrightarrow{\quad} \circ \end{array}$$

**Definition** We say that a morphism  $\psi : B_1 \rightarrow B_2$  of crystals is *strict* if for all  $b \in B_1$  and for all  $i \in I$ ,  $\psi(\tilde{e}_i b) = \tilde{e}_i \psi(b)$  and  $\psi(\tilde{f}_i b) = \tilde{f}_i \psi(b)$ .

**Remark** It will later be useful to restrict the morphisms in our category to strict ones so we may use the following result, the proof of which is discussed in [5].

**Lemma 1.3** (Schur's Lemma for Strict Morphisms,[5]). *A nonzero strict morphism between irreducible crystals arising from integrable  $U_q(\mathfrak{g})$ -modules is an isomorphism.*

**Proposition 1.4.** *All finite irreducible  $\mathfrak{sl}_2$  crystals are of the form  $B(n) \otimes T_\lambda$  for some  $n \in \mathbb{N}, \lambda \in \mathbb{Z}$ .*

$$B(l) \otimes T_{\lambda-l} \rightarrow C, \quad x^i y^j \otimes t_{\lambda-l} \mapsto \tilde{f}^i c_0$$

The category of crystals is endowed with a tensor product as in [7], arising naturally from the one constructed for crystals arising from integrable  $U_q(\mathfrak{g})$ -modules by Kashiwara.

$$\begin{aligned} \tilde{e}_i(b_1 \otimes b_2) &= \begin{cases} \tilde{e}_i b_1 \otimes b_2 & \text{if } \phi_i(b_1) \geq \varepsilon_i(b_2) \\ b_1 \otimes \tilde{e}_i b_2 & \text{if } \phi_i(b_1) < \varepsilon_i(b_2), \end{cases} \\ \tilde{f}_i(b_1 \otimes b_2) &= \begin{cases} \tilde{f}_i b_1 \otimes b_2 & \text{if } \phi_i(b_1) > \varepsilon_i(b_2) \\ b_1 \otimes \tilde{f}_i b_2 & \text{if } \phi_i(b_1) \leq \varepsilon_i(b_2), \end{cases} \\ \varepsilon_i(b_1 \otimes b_2) &= \max(\varepsilon_i(b_1), \varepsilon_i(b_2) - \lambda_i(\text{wt}(b_1))), \\ \phi_i(b_1 \otimes b_2) &= \max(\phi_i(b_1) + \lambda_i(\text{wt}(b_1)), \phi_i(b_2)), \\ \text{wt}(b_1 \otimes b_2) &= \text{wt}(b_1) + \text{wt}(b_2). \end{aligned}$$

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products of crystals. In the case of  $\mathfrak{sl}_2$ , this has an explicit description. If  $\zeta : B \rightarrow B$  exchanges the highest and lowest weight elements of a crystal, essentially reversing the crystal graph, then  $b \otimes b' \mapsto \zeta(\zeta(b') \otimes \zeta(b))$ .

**1.3. The Clebsch-Gordan Formula.** In the case of  $\mathfrak{sl}_2$ , since  $U(\mathfrak{sl}_2)$  forms a bialgebra,  $V(n) \otimes V(m)$  is a  $U(\mathfrak{sl}_2)$ -module for every pair  $m, n \in \mathbb{N}$  and so must decompose into simple modules. This decomposition is determined explicitly by the Clebsch-Gordan Formula:

**Lemma 1.5.** (The Clebsch-Gordan Formula) [11, p. 105 & p. 175] *For all  $n, m \in \mathbb{N}$ , there is an isomorphism of  $U(\mathfrak{sl}_2)$ -representations*

$$V(n) \otimes V(m) \cong \bigoplus_{k=0}^{\min(m,n)} V(m+n-2k).$$

*Furthermore, the same Clebsch-Gordan formula holds in the case of  $U_q(\mathfrak{sl}_2)$ -representations.*

We can see a similar Clebsch-Gordan decomposition for  $\mathfrak{sl}_2$  crystals, the proof of which is simple and explicit.

**Proposition 1.6.** (Clebsch-Gordan for  $\mathfrak{sl}_2$  Crystals) *For all  $m, n \in \mathbb{N}$  there is an isomorphism*

$$B(m) \otimes B(n) \cong \bigsqcup_{k=0}^{\min(m,n)} B(m+n-2k) = \bigsqcup_{\substack{|m-n| < k < m+n, \\ m+n \equiv k \pmod{2}}} B(k).$$

*Proof.* Suppose  $m \geq n$ . Consider  $y^m \otimes x^k y^{n-k} \in B(m) \otimes B(n)$  for  $k = 0, 1, \dots, n$ . From the above we have:

$$\tilde{e}(y^m \otimes x^k y^{n-k}) = 0$$

$$\tilde{f}^i(y^m \otimes x^k y^{n-k}) = \begin{cases} x^i y^{m-i} \otimes x^k y^{n-k} & \text{if } i < m-k \\ x^{m-k} y^k \otimes x^{i+2k-m} y^{m+n-2k-i} & \text{if } m-k \leq i \leq (m+n)-2k \\ 0 & \text{otherwise} \end{cases}$$

Thus  $y^m \otimes x^k y^{n-k}$  is a highest weight vector in a string of length  $(m+n)-2k$ , so we have an inclusion of crystals  $\bigsqcup_{k=0}^n B(m+n-2k) \hookrightarrow B(m) \otimes B(n)$ . By the pigeonhole principal, this map of crystals is also surjective, hence is an isomorphism. Similarly, if  $m < n$ , we may consider  $y^m \otimes x^k y^{n-k} \in B(m) \otimes B(n)$  for  $k = 0, 1, \dots, m$  and we obtain the same result.  $\square$

As a result of the above, we have  $B(1) \otimes B(n) \cong B(n-1) \sqcup B(n+1)$  for  $n \geq 1$ . This gives us the following:

**Proposition 1.7.** *There is an embedding*

$$B(n) \hookrightarrow B(1)^{\otimes n}$$

for all  $n \in \mathbb{N}$ , which is given by

$$x^i y^j \mapsto \overbrace{x \otimes x \otimes \dots \otimes x}^{i \text{ times}} \otimes \overbrace{y \otimes \dots \otimes y}^{j \text{ times}}.$$

**Remark** Using this Clebsch-Gordan decomposition for crystals we may also compute Henriques' and Kamnitzer's commuter of crystals explicitly.

**Proposition 1.8.** *Let  $x^i y^j \otimes x^r y^s \in B(n) \otimes B(m)$ . Then the commuter of crystals in [5] gives*

$$\sigma_{B(n) \otimes B(m)} : x^i y^j \otimes x^r y^s \mapsto \begin{cases} x^{i+r-j} y^{s+j-i} \otimes x^j y^i & \text{if } j \leq r, i \leq s \\ x^{s+r-j} y^j \otimes x^{i+j-s} y^s & \text{if } j \leq r, i > s \\ x^i y^{r+s-i} \otimes x^r y^{i+j-r} & \text{if } j > r, i \leq s \\ x^s y^r \otimes x^{i+r-s} y^{j+s-r} & \text{if } j > r, i > s. \end{cases}$$

*Proof.* By definition,

$$\sigma_{B(n) \otimes B(m)}(x^i y^j \otimes x^r y^s) = \zeta(\zeta(x^r y^s) \otimes \zeta(x^i y^j)) = \zeta(x^s y^r \otimes x^j y^i).$$

If  $j \leq r$  then  $x^s y^r \otimes x^j y^i$  corresponds to  $x^s y^{i+r-j}$  in an isomorphic copy of  $B(n+m-2j)$ . Then, applying  $\zeta$  we obtain  $x^{i+r-j} y^s \in B(n+m-2j)$ . If  $i \leq s$  then this corresponds to  $x^{i+r-j} y^{s+j-i} \otimes x^j y^i$  in  $B(m) \otimes B(n)$ , and if  $i > s$  then this corresponds to  $x^{s+r-j} y^j \otimes x^{i+j-s} y^s$  in  $B(m) \otimes B(n)$ . Now if  $j > r$  then  $x^s y^r \otimes x^j y^i$  corresponds to  $x^{s+j-r} y^i$  in a copy of  $B(n+m-2r)$ . Then  $\zeta(x^{s+j-r} y^i) = x^i y^{s+j-r} \in B(n+m-2r)$ . So if  $i \leq s$  this corresponds to  $x^i y^{r+s-i} \otimes x^r y^{i+j-r}$  and if  $i > s$  then this corresponds to  $x^s y^r \otimes x^{i+r-s} y^{j+s-r}$ .  $\square$

More generally, we have a way of determining how the tensor product of such crystals decomposes as proven in Kashiwara's paper [8].

**Proposition 1.9.** (Decomposition of Tensor Product of Crystals, [8]) *There is an isomorphism of crystals*

$$B(\alpha) \otimes B(\beta) \cong \bigsqcup B(\alpha + wt(b))$$

where the disjoint union ranges over all  $b \in B(\beta)$  such that  $\varepsilon_i(b) \leq \lambda_i(\alpha)$  for all  $i \in I$ . Most importantly,  $\varepsilon_i(u_\beta) = 0 \leq \lambda_i(\alpha)$  for each  $i$ , so  $B(\alpha + \beta)$  appears as a term in this decomposition since  $\alpha \in \Phi_+ = \{\alpha \in \Phi \mid \lambda_i(\alpha) \geq 0 \text{ for any } i \in I\}$ .

**Definition** For a crystal  $B$ , we may construct a crystal  $B^\vee$ , as in [8], obtained by reversing arrows in its crystal graph. More precisely,  $B^\vee = \{b^\vee \mid b \in B\}$  with  $\tilde{e}_i(b^\vee) = (\tilde{f}_i b)^\vee$ ,  $\tilde{f}_i(b^\vee) = (\tilde{e}_i b)^\vee$ ,  $wt(b^\vee) = -wt(b)$ ,  $\varepsilon_i(b^\vee) = \phi_i(b)$  and  $\phi_i(b^\vee) = \varepsilon_i(b)$ . Then, for  $\alpha \in \Phi$  we will define  $B(-\alpha) := B(\alpha)^\vee$ , and we will use this as our notion of dual  $B(\alpha)^*$  to  $B(\alpha)$ .

**Remark** Both  $B(\alpha) \otimes B(-\alpha)$  and  $B(-\alpha) \otimes B(\alpha)$  contain in their decompositions the trivial crystal, respectively  $\{u_\alpha \otimes u_{-\alpha}\}$  and  $\{u_{-\alpha} \otimes u_\alpha\}$  where  $u_{-\alpha} = u_\alpha^\vee$ . For crystals  $B_1, B_2$  we can see that  $(B_1 \otimes B_2)^\vee \cong B_2^\vee \otimes B_1^\vee$ , thus

we may deduce that  $B(-(\alpha + \beta))$  appears as a term in the decomposition of  $B(-\alpha) \otimes B(-\beta)$ . In the case of  $\mathfrak{sl}_2$ , we simply have  $B(n)^\vee \cong B(n)$ .

**1.4. Multiplication on the  $\mathfrak{sl}_2$ -crystal  $\bigsqcup_{n \in \mathbb{N}} B(n)$ .** As a result of the Clebsch-Gordan formula, we may define a multiplication on the  $U(\mathfrak{sl}_2)$ -module  $\oplus_{n \in \mathbb{N}} V(n)^*$  as follows. For  $n \in \mathbb{N}$  we consider the simple  $(n+1)$ -dimensional  $U(\mathfrak{sl}_2)$ -module  $V(n)$  with usual basis  $\{u_k^{(n)}\}_{0 \leq k \leq n}$  of  $h$ -eigenvectors (see [11, p. 101]), and its dual  $V(n)^*$  with dual basis  $\{\tilde{u}_k^{(n)}\}_{0 \leq k \leq n}$ . By the Clebsch-Gordan formula, we have an injection

$$\phi : V(m+n) \hookrightarrow \bigoplus_{k=0}^{\min(m,n)} V(m+n-2k) \cong V(m) \otimes V(n), \quad u_0^{(m+n)} \mapsto u_0^{(m)} \otimes u_0^{(n)}.$$

This gives a transpose map

$$\phi^* = \mu_1 : V(n)^* \otimes V(m)^* \rightarrow (V(n) \otimes V(m))^* \rightarrow V(n+m)^*,$$

which can be explicitly computed as

$$\tilde{u}_i^{(n)} \otimes \tilde{u}_j^{(m)} \mapsto \tilde{u}_{i+j}^{(m+n)}.$$

If we let  $x^i y^j$  denote  $\tilde{u}_i^{(i+j)}$  as above for  $i, j \in \mathbb{N}$ , that is,  $\tilde{u}_i^{(n)} = x^i y^{n-i}$ , then this multiplication becomes

$$\mu_1 : x^{i_1} y^{j_1} \otimes x^{i_2} y^{j_2} \mapsto x^{i_1+i_2} y^{j_1+j_2}.$$

This multiplication agrees with the multiplication on the affine plane,  $k[x, y]$ , over our field  $k$ .

Similarly, for  $n \in \mathbb{N}$ , we may consider the simple  $U_q(\mathfrak{sl}_2)$ -module  $V(n)$  and its dual  $V(n)^*$ . Again, we give these respectively the usual basis  $\{u_k^{(n)}\}_{0 \leq k \leq n}$  of  $t$ -eigenvectors (see [11, p. 127]) and dual basis  $\{\tilde{u}_k^{(n)}\}_{0 \leq k \leq n}$ . The analogous map

$$\mu_q : V(n)^* \otimes V(m)^* \rightarrow (V(n) \otimes V(m))^* \rightarrow V(n+m)^*$$

can be expressed

$$\tilde{u}_i^{(m)} \otimes \tilde{u}_j^{(n)} \mapsto q^{-(m-i)j} \tilde{u}_{i+j}^{(m+n)}.$$

If we again let  $x^i y^j$  denote  $\tilde{u}_i^{(i+j)}$  for  $i, j \in \mathbb{N}$  this multiplication becomes

$$\mu_q : x^{i_1} y^{j_1} \otimes x^{i_2} y^{j_2} \mapsto q^{-j_1 i_2} x^{i_1+i_2} y^{j_1+j_2}.$$

Similarly to before, this multiplication agrees with that of the quantum affine plane,  $k_q[x, y] := k\langle x, y \rangle / (xy - qyx)$ . When  $q = 1$ , this agrees with  $\mu_1$  as before.

Returning to crystals, from the corresponding Clebsch Gordan formula for  $\mathfrak{sl}_2$  crystals we have two maps, for  $n \in \mathbb{N}$ ,

$$B(0) \hookrightarrow B(n)^* \otimes B(n), B(n) \otimes B(n)^* \twoheadrightarrow B(0).$$

If we look at the chain of maps

$$B(n) \cong B(n) \otimes B(0) \hookrightarrow B(n) \otimes B(n)^* \otimes B(n) \twoheadrightarrow B(0) \otimes B(n) \cong B(n)$$

$$x^i y^j \mapsto x^i y^j \otimes x^0 y^0 \mapsto x^i y^j \otimes y^n \otimes x^n \mapsto 0 \mapsto 0$$

we see that for  $n \neq 0$ , this composition is the zero map and so is not the identity on  $B(n)$ . We see a similar result with the composition

$$B(n) \cong B(0) \otimes B(n) \hookrightarrow B(n) \otimes B(n)^* \otimes B(n) \twoheadrightarrow B(n) \otimes B(0) \cong B(n).$$

Thus we do not have duality between crystals  $B(n)$  and  $B(n)^*$ . We can, however, still use Clebsh Gordan to define a multiplication map on  $\bigsqcup_{n \in \mathbb{N}} B(n)$  as in the non crystal case.

We define the multiplication map  $\mu_0$  by

$$\begin{aligned} \mu_0 : \bigsqcup_{n \in \mathbb{N}} B(n) \otimes \bigsqcup_{n \in \mathbb{N}} B(n) &\rightarrow \bigsqcup_{n \in \mathbb{N}} B(n) \\ B(n) \otimes B(m) &\cong \bigsqcup_{k=0}^{\min(m,n)} B(m+n-2k) \twoheadrightarrow B(m+n). \end{aligned}$$

Which, explicitly, gives the map

$$\mu_0 : x^i y^j \otimes x^r y^s \mapsto \begin{cases} x^{i+r} y^s & \text{if } j = 0 \\ x^i y^{j+s} & \text{if } r = 0 \\ 0 & \text{if } j \neq 0 \neq r. \end{cases}$$

This is easily seen when we look at the crystal graph of  $B(n) \otimes B(m)$  seen earlier. Then if we identify  $B(n)$  with  $B(n)^\vee$  then the multiplication on  $\bigsqcup_{n \in \mathbb{N}} B(n)^\vee$  is

$$(x^i y^j)^\vee \cdot (x^r y^s)^\vee = \begin{cases} (x^r y^{j+s})^\vee & \text{if } i = 0 \\ (x^{i+r} y^j)^\vee & \text{if } s = 0 \\ 0 & \text{if } i \neq 0 \neq s. \end{cases}$$

The if we denote  $y^i x^j = (x^j y^i)^\vee$ , this agrees with the multiplication on the monic monomials in the quotient space  $k_0[x, y] := k\langle x, y \rangle / (xy)$ . This can be thought of as the limit as  $q$  tends to 0 of  $k_q[x, y]$ , which arises in the quantum case.

We may also define the unit map  $\eta$  by

$$\eta : B(0) \hookrightarrow \bigsqcup_{n \in \mathbb{N}} B(n), \quad x^0 y^0 \mapsto x^0 y^0$$

**Proposition 1.10.** *The crystal  $\bigsqcup_{n \in \mathbb{N}} B(n)$  forms an algebra object in  $\text{Crys}$  given by the triple  $(\bigsqcup_{n \in \mathbb{N}} B(n), \mu_0, \eta)$ .*

**Remark** Of course, this result is easily generalised. The projections  $B(-\alpha) \otimes B(-\beta) \twoheadrightarrow B(-(\alpha + \beta))$  induce an algebra structure on  $\bigsqcup_{\alpha \in \Phi_+} B(-\alpha)$ .

## 2. THE CRYSTAL COALGEBRA $\mathcal{B}$

Recall the definition of the quantum co-ordinate ring  $A_q(\mathfrak{g})$  from the first section. It is well known that its comodules are precisely the representations of  $U_q(\mathfrak{g})$  in  $\mathcal{O}_{\mathfrak{g}}$ . The focus of this paper is to investigate whether a similar result is true in the setting of crystal bases. We will consider the crystal equivalent of this module,

$$\mathcal{B} := \bigsqcup_{\alpha \in \Phi^+} B(\alpha) \otimes B(\alpha)^*,$$

with the hope that we may exhibit some crystals as an analogue of comodules over a bialgebra-like structure.

### 2.1. The Coalgebra $\mathcal{B}$ .

**Definition** Let

$$\mathcal{B} := \bigsqcup_{\alpha \in \Phi_+} B(\alpha) \otimes B(-\alpha).$$

**Remark** We have a map

$$\iota : B(0) \hookrightarrow B(-\alpha) \otimes B(\alpha), \quad b_0 \mapsto b_{-\alpha} \otimes b_{\alpha}$$

allowing us to define

$$\Delta : \mathcal{B} \rightarrow \mathcal{B} \otimes \mathcal{B}$$

given by the compositions

$$\begin{aligned} B(\alpha) \otimes B(-\alpha) &\rightarrow B(\alpha) \otimes B(0) \otimes B(-\alpha) \\ &\rightarrow B(\alpha) \otimes B(-\alpha) \otimes B(\alpha) \otimes B(-\alpha). \end{aligned}$$

Coassociativity of this map is clear. For the comultiplication to have a counit  $\varepsilon$ , we would need the following commutative diagram:

$$\begin{array}{ccccc} B(0) \otimes \mathcal{B} & \xleftarrow{\varepsilon \otimes \text{id}} & \mathcal{B} \otimes \mathcal{B} & \xrightarrow{\text{id} \otimes \varepsilon} & \mathcal{B} \otimes B(0) \\ & \nwarrow \cong & \uparrow \Delta & \nearrow \cong & \\ & & \mathcal{B} & & \end{array}$$

In the case of  $\mathfrak{sl}_2$ , we would require that in general

$$\varepsilon(x^i y^j \otimes y^n) \otimes x^n \otimes x^r y^s = x^0 y^0 \otimes x^i y^j \otimes x^r y^s,$$

$$x^i y^j \otimes y^n \otimes \varepsilon(x^n \otimes x^r y^s) = x^i y^j \otimes x^r y^s \otimes x^0 y^0,$$

which is not possible for any  $\varepsilon$ .

**Proposition 2.1.** *There is a non-counital coalgebra structure on  $\mathcal{B}$  in the category  $\text{Crys}$ .*

## 2.2. $\mathcal{B}$ Comodules.

**Definition** By a  $\mathcal{B}$ -comodule, we mean a pointed set  $C$  with a coaction map  $\Delta_C : C \rightarrow \mathcal{B} \otimes C$  satisfying the appropriate commutative diagrams in the category of pointed sets. In the absence of a counit we also require that  $\Delta_C(c) \neq 0$  for all  $c \in C$ . We may refer to these as the comodules of  $\mathcal{B}$  in the category of pointed sets.

**Proposition 2.2.** *All crystals that arise from integrable  $\mathfrak{g}$ -modules have a  $\mathcal{B}$ -comodule structure.*

*Proof.* With respect to this coalgebra crystal, we may exhibit  $B(\alpha)$  as comodules, which will give our result. We define the coaction

$$\begin{aligned} \Delta_\alpha : B(\alpha) &\cong B(\alpha) \otimes B(0) \hookrightarrow B(\alpha) \otimes B(-\alpha) \otimes B(\alpha) \hookrightarrow \mathcal{B} \otimes B(\alpha) \\ b &\mapsto (b \otimes u_{-\alpha}) \otimes u_\alpha. \end{aligned}$$

□

In the case of  $\mathfrak{sl}_2$ , we can say a little more

**Corollary 2.3.** *All finite irreducible  $\mathfrak{sl}_2$  crystals, and hence all  $\mathfrak{sl}_2$  crystals whose connected components are finite, are comodules over  $\mathcal{B}$ .*

*Proof.* If we again let  $T_\lambda = \{t_\lambda\}$  be the singleton crystal with weight  $\lambda \in \mathbb{Z}$  then we have a comodule structure on  $B(n) \otimes T_\lambda$  extending  $\Delta_n$ ,

$$\Delta_{n,\lambda} : B(n) \otimes T_\lambda \rightarrow \mathcal{B} \otimes B(n) \otimes T_\lambda, \quad x^i y^j \otimes t_\lambda \mapsto (x^i y^j \otimes y^n) \otimes x^n \otimes t_\lambda.$$

As we have seen, in the case of  $\mathfrak{sl}_2$  crystals, all finite irreducible crystals are of the form  $B(n) \otimes T_\lambda$  for some integer  $\lambda$ . Thus we can give a comodule structure to any finite irreducible crystal. □

**Remark** It is clear, however, that the coaction given above does not distinguish between the different weights  $\lambda$ .

**Remark** Suppose  $C$  is a  $\mathcal{B}$ -comodule with coaction  $\Delta_C : C \rightarrow \mathcal{B} \otimes C$ . Let  $c \in C$ , and suppose  $\Delta_C(c) = (b_1 \otimes b_2) \otimes c'$  for  $c' \in C$ ,  $b_1 \otimes b_2 \in \mathcal{B}$ . Suppose further that  $\Delta_C(c') = (b'_1 \otimes b'_2) \otimes c''$  for  $c'' \in C$ ,  $b'_1 \otimes b'_2 \in \mathcal{B}$ . Then, by definition, we have, for some  $\alpha \in \Phi_+$ ,

$$b_1 \otimes u_{-\alpha} \otimes u_\alpha \otimes b_2 \otimes c' = (\Delta \otimes \text{id}) \circ \Delta_C(c) = (\text{id} \otimes \Delta_C) \circ \Delta_C(c) = b_1 \otimes b_2 \otimes b'_1 \otimes b'_2 \otimes c''$$

So we know that  $b_2 = u_{-\alpha} = b'_2$ ,  $b'_1 = u_\alpha$  and  $c' = c''$ . So, for a general  $c \in C$ , there is  $c' \in C$ ,  $n \in \mathbb{N}$  and  $b \in B(n)$  such that

$$\Delta_C(c) = b \otimes u_{-\alpha} \otimes c', \quad \Delta(c') = u_\alpha \otimes u_{-\alpha} \otimes c'.$$

For a comodule  $C$ , denote by  $\hat{C}$  the subcomodule

$$\hat{C} = \{c \in C \mid \Delta_C(c) = u_\alpha \otimes u_{-\alpha} \otimes c \text{ for some } \alpha \in \Phi\}.$$

For each  $c \in \hat{C}$ , the set  $\{c\}$  is a simple subcomodule, and every nontrivial subcomodule of  $C$  contains one of these singleton subcomodules. Thus the simple subcomodules are precisely these singleton subcomodules  $\{c\}$  for  $c \in \hat{C}$ . We again see that  $\mathcal{B}$  is not semisimple. In this context, we mean that

comodules such as these are semisimple if they are a union of irreducible comodules. This contrasts with the quantum coordinate algebra  $A_q(\mathfrak{g})$ , which is semisimple as a coalgebra. Thus it appears that we may not be able to classify all  $\mathcal{B}$ -comodules in the same way.

Indeed, consider again the case of  $\mathfrak{sl}_2$ . Suppose that the categories of comodules of  $\mathcal{B}$  and of crystals coincide. Consider a comodule  $C = \{a, b\}$  with coaction defined by  $\Delta_C(a) = y \otimes y \otimes b$ ,  $\Delta_C(b) = x \otimes y \otimes b$ . This has subcomodule  $\{b\}$ , and so by our assumption both  $C$  and  $\{b\}$  would have crystal structures compatible with the inclusion. Then they are disjoint unions of connected components with respect to their crystal graphs, so  $\{a\}$  must also have an appropriate crystal structure, and hence must also be a subcomodule. But this is not true.

**Remark** The above maps are (strict) morphisms of crystals, so in fact such crystals are comodules of  $\mathcal{B}$  in the subcategory of crystals whose irreducible components are finite. Of course, it is then a trivial fact that the comodules of  $\mathcal{B}$  in this subcategory of crystals form the entire category, and so give a classification of the category.

### 2.3. Multiplication on $\mathcal{B}$ .

**Remark** We also have maps

$$\begin{aligned} \vartheta & : B(\alpha) \otimes B(\beta) \rightarrow \bigsqcup_{\gamma} B(\gamma), \\ \vartheta^{\vee} & : B(-\beta) \otimes B(-\alpha) \rightarrow \bigsqcup_{\gamma'} B(-\gamma') \text{ and} \\ & B(0) \cong B(0) \otimes B(-0), \end{aligned}$$

where  $\vartheta$  and  $\vartheta'$  decompose the tensor product into irreducible crystals. Hence we may define unit and multiplication maps

$$\eta : B(0) \cong B(0) \otimes B(-0) \hookrightarrow \bigsqcup_{\alpha \in \Phi_+} B(\alpha) \otimes B(-\alpha) = \mathcal{B},$$

and

$$\mu : \mathcal{B} \otimes \mathcal{B} \rightarrow \mathcal{B}$$

given by

$$\begin{aligned} B(\alpha) \otimes B(-\alpha) \otimes B(\beta) \otimes B(-\beta) & \rightarrow B(\alpha) \otimes B(\beta) \otimes B(-\beta) \otimes B(-\alpha) \\ & \rightarrow (\bigsqcup_{\gamma} B(\gamma)) \otimes (\bigsqcup_{\gamma'} B(-\gamma')) \\ & \rightarrow \bigsqcup_{\gamma} B(\gamma) \otimes B(-\gamma) \\ & \rightarrow \mathcal{B}, \end{aligned}$$

In the definition of multiplication map, the first arrow interchanges the positions of the crystals in the product of sets, the second uses  $\vartheta$  and  $\vartheta'$  to decompose these tensor products. The irreducible components in the decomposition of the crystal  $B(\alpha) \otimes B(\beta)$  correspond directly to irreducible components in the decomposition of the crystal  $B(-\beta) \otimes B(-\alpha) = (B(\alpha) \otimes B(\beta))^{\vee}$ , and the third projects onto matching pairs of these. It is clear that these maps give an algebra structure on  $\mathcal{B}$ .

**Proposition 2.4.** *There is a non-counital bialgebra structure on  $\mathcal{B}$ .*

*Proof.* It is easy to see that we have the compatibility required for  $\mathcal{B}$  to be a (non-counital) bialgebra. Indeed, for  $b \otimes b' \in B(\alpha)$ ,  $b'' \otimes b''' \in B(\alpha)$

$$\begin{array}{ccc}
(b \otimes b'^\vee) \otimes (b'' \otimes b'''^\vee) & \xrightarrow{\Delta \otimes \Delta} & (b \otimes b_\alpha^\vee) \otimes (b_\alpha \otimes b'^\vee) \otimes (b'' \otimes b_\beta^\vee) \otimes (b_\beta \otimes b'''^\vee) \\
\downarrow \mu & & \downarrow \text{id} \otimes \tau \otimes \text{id} \\
& & (b \otimes b_\alpha^\vee) \otimes (b'' \otimes b_\beta^\vee) \otimes (b_\alpha \otimes b'^\vee) \otimes (b_\beta \otimes b'''^\vee) \\
& & \downarrow \mu \otimes \mu \\
(b \cdot b'') \otimes (b''' \cdot b')^\vee & \xrightarrow{\Delta} & ((b \cdot b'') \otimes b_{\alpha+\beta}^\vee) \otimes (b_{\alpha+\beta} \otimes (b''' \cdot b')^\vee)
\end{array}$$

where we denote by  $b \cdot b'$  the corresponding element of  $B(\alpha) \otimes B(\beta)$  in the decomposition into irreducible crystals, and we understand that  $c \otimes d^\vee$  is 0 if  $c, d$  lie in different irreducible crystals.  $\square$

**Remark** There is also an injection of algebras  $\bigsqcup_{\alpha \in \Phi_+} B(-\alpha) \hookrightarrow \mathcal{B}$  given by  $b^\vee \rightarrow b_\alpha \otimes b^\vee$  for  $b \in B(\alpha)$ , however this is not a morphism of crystals.

**Remark** Note that the multiplication on  $\mathcal{B}$  allows us to give a comodule structure to  $B(\alpha) \otimes B(\beta)$  via

$$\begin{aligned}
\Delta_{\alpha,\beta} : B(\alpha) \otimes B(\beta) & \xrightarrow{\Delta_\alpha \otimes \Delta_\beta} \mathcal{B} \otimes B(\alpha) \otimes \mathcal{B} \otimes B(\beta) \\
& \xrightarrow{\text{id} \otimes \tau \otimes \text{id}} \mathcal{B} \otimes \mathcal{B} \otimes B(\alpha) \otimes B(\beta) \rightarrow \mathcal{B} \otimes B(\alpha) \otimes B(\beta).
\end{aligned}$$

But then  $\Delta_{\alpha,\beta}(b \otimes b') = (b \cdot b') \otimes b_{\alpha+\beta} \otimes (b_\alpha \otimes b_\beta)$  if  $b \cdot b' \in B(\alpha + \beta)$  but is 0 otherwise. Hence this is not the same coaction as the one induced by the decomposition of the tensor product, except from on the copy of  $B(\alpha + \beta)$  in  $B(\alpha) \otimes B(\beta)$ .

**Remark** Unfortunately, this multiplication map is not given by a morphism of crystals. This is for much the same reason that the multiplication on  $A_q(\mathfrak{g})$  is not a morphism of  $U_q(\mathfrak{g})$  modules. We may ask if we can define an alternative multiplication,  $\mu'$  say, on  $\mathcal{B}$  as before but using the commutator of crystals  $\sigma$  from [5] instead of the usual twist of pointed sets  $\tau$ . This would raise several issues, since  $\sigma$  does not satisfy the usual braiding properties that we would want. Also, such a multiplication would not be compatible with the comultiplication  $\Delta$ . In the case of  $\mathfrak{sl}_2$ , where these can be computed explicitly, if we let  $b = x^n \otimes y^n$ ,  $b' = x^m \otimes y^m \in \mathcal{B}$  for  $m, n \in \mathbb{N}$  then we can



compute that

$$\begin{aligned} \mu'_{\mathcal{B} \otimes \mathcal{B}} \circ (\Delta \otimes \Delta) : b \otimes b' &\mapsto \begin{cases} x^n y^m \otimes x^n y^m \otimes x^n y^m \otimes x^n y^m & \text{if } m \leq n \\ x^n y^m \otimes x^m y^n \otimes x^n y^m \otimes x^m y^n & \text{if } m \geq n \end{cases} \\ \mu' \circ \Delta : b \otimes b' &\mapsto \begin{cases} x^n y^m \otimes y^{n+m} \otimes x^{n+m} \otimes x^n y^m & \text{if } m \leq n \\ x^m y^n \otimes y^{n+m} \otimes x^{n+m} \otimes x^n y^m & \text{if } m \geq n. \end{cases} \end{aligned}$$

These maps do not agree for strictly positive  $n$  and  $m$ , so we do not obtain the desired compatibility.

## Part 2. A Functorial Approach to Crystals

### 3. THE BARR-BECK THEOREM

**3.1. Monads and Comonads.** We begin by recalling definitions of monads and comonads, the generalised notions of algebras and coalgebras in the setting of functors on categories. For more details see Borceaux's *Handbook of Categorical Algebra 2* [2, p. 189-197].

**Definition** A *monad* on a category  $\mathcal{C}$  is a triple  $\mathbb{T} = (T, \eta, \mu)$  where  $T : \mathcal{C} \rightarrow \mathcal{C}$  is a functor and  $\eta : \text{id}_{\mathcal{C}} \Rightarrow T$ ,  $\mu : T \circ T \Rightarrow T$  are natural transformations satisfying the usual associativity and unit constraints as for an algebra.

As algebras have modules, monads have what are known as *algebras* over them.

**Definition** For a category  $\mathcal{C}$  with monad  $\mathbb{T} = (T, \eta, \mu)$  as above, an *algebra* on this monad is a pair  $(C, \xi)$  where  $C$  is an object in the category and  $\xi : T(C) \rightarrow C$  is a morphism in the category satisfying appropriate compatibility requirements. A *morphism of algebras*  $f : (C, \xi) \rightarrow (C', \xi')$  is a morphism  $f : C \rightarrow C'$  in the category such that  $f \circ \xi = \xi' \circ T(f)$ . These algebras in  $\mathcal{C}$  over a monad  $\mathbb{T}$  form a category, denoted  $\mathcal{C}^{\mathbb{T}}$ , known as the *Eilenberg-Moore category* of the monad.

**Definition** Dually, we define a *comonad* on a category  $\mathcal{C}$  as a triple  $\mathbb{U} = (U, \varepsilon, \Delta)$ , where  $U : \mathcal{C} \rightarrow \mathcal{C}$  is a functor and  $\varepsilon : U \Rightarrow \text{id}_{\mathcal{C}}$ ,  $\Delta : U \Rightarrow U \circ U$  are natural transformations satisfying the dual compatibility requirements. A *coalgebra* on this monad to be a pair  $(D, \zeta)$  where  $D$  is an object in the category and  $\zeta : D \rightarrow U(D)$  is an appropriate morphism in the category, and a *morphism of coalgebras*  $g : (D, \zeta) \rightarrow (D', \zeta')$  is a morphism  $g : D \rightarrow D'$  in the category such that  $U(g) \circ \zeta = \zeta' \circ g$ . These coalgebras in  $\mathcal{C}$  over a comonad  $\mathbb{U}$  form a category, which we shall denote  $\mathcal{C}_{\mathbb{U}}$ .

**Remark** Suppose we have a pair of adjoint functors  $F : \mathcal{C} \rightarrow \mathcal{D}$ ,  $G : \mathcal{D} \rightarrow \mathcal{C}$  with  $F \dashv G$ . Let  $\eta : \text{id}_{\mathcal{C}} \Rightarrow G \circ F$  be the unit of the adjunction and  $\varepsilon : F \circ G \Rightarrow \text{id}_{\mathcal{D}}$  the counit. Then  $\mathbb{T} = (T := G \circ F, \eta, \mu)$  defines a monad where  $\mu$  is the horizontal composition  $\mu = \text{id}_G * \varepsilon * \text{id}_F : GF GF \Rightarrow G \circ \text{id}_{\mathcal{D}} \circ F = GF$ . Similarly,  $\mathbb{U} = (U := F \circ G, \varepsilon, \Delta)$  forms a comonad where  $\Delta := \text{id}_F * \eta * \text{id}_G$ . Furthermore, we have *comparison functors*  $K^{\mathbb{T}} : \mathcal{D} \rightarrow \mathcal{C}^{\mathbb{T}}$ ,  $J_{\mathbb{U}} : \mathcal{C} \rightarrow \mathcal{D}_{\mathbb{U}}$

defined, respectively, by

$$\begin{aligned} K^{\mathbb{T}}(A) &= (G(A), G(\varepsilon_A)) \\ K^{\mathbb{T}}(f) &= G(f) \\ J_{\mathbb{U}}(B) &= (F(B), F(\eta_B)) \\ J_{\mathbb{U}}(g) &= F(g) \end{aligned}$$

for all objects  $A$  in  $\mathcal{D}$  and  $B$  in  $\mathcal{C}$  and for all morphisms  $f$  in  $\mathcal{D}$  and  $g$  in  $\mathcal{C}$ . These comparison functors allow us to give objects  $A$  in  $\mathcal{D}$  and objects  $B$  in  $\mathcal{C}$  the respective structures of algebras over  $\mathbb{T}$  and coalgebras over  $\mathbb{U}$ .

### 3.2. The Barr-Beck Theorem.

**Definition** A functor  $G : \mathcal{D} \rightarrow \mathcal{C}$  is called *monadic* if there exists a monad  $\mathbb{T} = (T, \eta, \mu)$  on  $\mathcal{C}$  and an equivalence of categories  $J : \mathcal{D} \rightarrow \mathcal{C}^{\mathbb{T}}$  such that  $F \circ J$  is isomorphic as a functor to  $G$ , where  $F : \mathcal{C}^{\mathbb{T}} \rightarrow \mathcal{C}$  is the forgetful functor. Again, see [2, p. 212] for more details. Equivalently ([17]), a functor  $G : \mathcal{D} \rightarrow \mathcal{C}$  is monadic if it has a left adjoint  $F : \mathcal{C} \rightarrow \mathcal{D}$ , and so the pair form a monad  $\mathbb{T} = (T := G \circ F, \eta, \mu)$  on  $\mathcal{C}$ , and if the comparison functor  $K^{\mathbb{T}} : \mathcal{D} \rightarrow \mathcal{C}^{\mathbb{T}}$  is an equivalence of categories.

**Definition** Dually, a functor  $F : \mathcal{C} \rightarrow \mathcal{D}$  is *comonadic* if it has a right adjoint  $G : \mathcal{D} \rightarrow \mathcal{C}$ , and so form a comonad  $\mathbb{U} = (U := F \circ G, \varepsilon, \Delta)$  on  $\mathcal{D}$ , and if the comparison functor  $J_{\mathbb{U}} : \mathcal{C} \rightarrow \mathcal{D}_{\mathbb{U}}$  is an equivalence of categories.

The following result, sometimes known as *Beck's Monadicity Theorem*, that gives criterion for when a functor is monadic (or comonadic).

**Theorem 3.1.** (The Barr-Beck Theorem [2, p. 212]) *A functor  $G : \mathcal{D} \rightarrow \mathcal{C}$  is monadic if and only if*

- i)  $G$  has a left adjoint  $F$ ;
- ii)  $G$  reflects isomorphisms. That is, if  $G(f)$  is an isomorphism then  $f$  is an isomorphism for all morphisms  $f$ ;
- iii) If a pair  $f, g : A \rightarrow B$  are morphisms in  $\mathcal{D}$  such that  $G(f), G(g)$  have a split coequaliser  $d : G(B) \rightarrow D$  in  $\mathcal{C}$  then  $f, g$  have a coequaliser  $c : B \rightarrow C$  in  $\mathcal{D}$  such that  $G(c) = d, G(C) = D$ .

A dual version of the Barr-Beck theorem then characterises comonadic functors as follows.

**Theorem 3.2.** *A functor  $F : \mathcal{C} \rightarrow \mathcal{D}$  is comonadic if and only if*

- i)  $F$  has a right adjoint  $G$ ;
- ii)  $F$  reflects isomorphisms;
- iii) If a pair  $f, g : A \rightarrow B$  are morphisms in  $\mathcal{C}$  such that  $F(f), F(g)$  have a split equaliser  $h : H \rightarrow F(A)$  in  $\mathcal{D}$  then  $f, g$  have an equaliser  $e : E \rightarrow A$  in  $\mathcal{C}$  such that  $F(e) = h, F(E) = H$ .

#### 4. CLASSIFICATION OF CRYSTALS

**4.1. The Crystal Functor.** From here we shall restrict our study to the crystals in  $Crys$  that arise from integrable  $\mathfrak{g}$ -modules. That is, crystals that are disjoint unions of  $B(\alpha)$  for  $\alpha \in \Phi_+$ . We will also consider only strict morphisms of crystals. We shall refer to this subcategory as  $Crys_{\mathfrak{g}}$ .

**Remark** Since all objects in the category of crystals are pointed sets with additional structure, there is a forgetful functor  $F : Crys_{\mathfrak{g}} \rightarrow Set_{\bullet}$  which sends a crystal to its underlying pointed set, forgetting this additional structure.

For us to apply Barr-Beck, we seek an adjunction between the category of crystals and the category of pointed sets using this forgetful functor.

**Definition** Consider the assignment

$$G : Ob(Set_{\bullet}) \rightarrow Ob(Crys_{\mathfrak{g}})$$

$$X \mapsto G(X) := \bigsqcup_{\alpha \in \Phi_+} \bigsqcup_{\substack{f \in \text{Hom}(FB(\alpha), X) \\ f \neq 0}} B(\alpha)_f$$

where  $B(\alpha)_f$  are distinct copies of  $B(\alpha)$  indexed by the functions  $f$ . Let functions  $\psi \in \text{Hom}_{Set_{\bullet}}(X, Y)$  be mapped to the morphisms  $G(\psi)$  gained from extending the isomorphisms  $B(\alpha)_f \rightarrow B(\alpha)_{\psi \circ f}$  (or mapping  $B(\alpha)_f$  to 0 if  $\psi \circ f = 0$ ). We can see that this describes a functor as  $G(id_X) = id_{G(X)}$  and for pointed sets  $X, Y, Z$  and morphisms  $\psi_1 : X \rightarrow Y$ ,  $\psi_2 : Y \rightarrow Z$ ,  $G(\psi_2 \circ \psi_1) = G(\psi_2) \circ G(\psi_1)$ .

**Proposition 4.1.** *There is an adjunction  $F \dashv G$  between the category of pointed sets and the category of crystals.*

*Proof.* Let  $\alpha \in \Phi_+$  and let  $Y$  be a set. Then  $B(\alpha)$  is a connected component so, by our crystal version of Shur's Lemma, a nonzero (strict) morphism of crystals  $f : B(\alpha) \rightarrow G(Y)$  is an isomorphism between  $B(\alpha)$  and an isomorphic copy  $B(\alpha)_{f'}$  for  $f' : FB(\alpha) \rightarrow Y$ . So such a map  $f : B(\alpha) \rightarrow G(Y)$  picks out a unique nonzero map of pointed sets  $f' : FB(\alpha) \rightarrow Y$ . This defines an isomorphism between the nonzero elements of  $\text{Hom}_{Crys_{\mathfrak{g}}}(B(\alpha), G(Y))$  and the nonzero elements of  $\text{Hom}_{Set_{\bullet}}(FB(\alpha), Y)$ . Thus we get an isomorphism

$$\text{Hom}_{Crys_{\mathfrak{g}}}(B(\alpha), G(Y)) \cong \text{Hom}_{Set_{\bullet}}(FB(\alpha), Y)$$

by sending the zero map to the zero map. For a general crystal  $X$ , let  $X_i$  be the irreducible components for  $i \in I$ , an indexing set. Then the  $X_i$  are isomorphic to  $B(\alpha_i)$  for some  $\alpha_i \in \Phi_+$ . So the isomorphisms  $\text{Hom}_{Crys_{\mathfrak{g}}}(X_i, G(Y)) \cong \text{Hom}_{Set_{\bullet}}(F(X_i), Y)$  extend to an isomorphism  $\rho_{X,Y} : \text{Hom}_{Crys_{\mathfrak{g}}}(X, G(Y)) \cong \text{Hom}_{Set_{\bullet}}(F(X), Y)$  since  $F$  commutes with disjoint unions. It is not hard to see that this isomorphism is natural.  $\square$

**4.2. Classifying Crystals.** In order to apply the Barr-Beck theorem to our pair of adjoint functors, we must check that the forgetful functor,  $F$ , reflects isomorphisms and preserves split equalisers. The fact that  $F$  reflects isomorphisms is apparent.

**Proposition 4.2.**  *$\text{Crys}_{\mathfrak{g}}$  has, and  $F$  preserves, all equalisers.*

*Proof.* Suppose we have parallel maps  $f, g : X \rightarrow Y$  in  $\text{Crys}_{\mathfrak{g}}$ . It is enough to check that  $f$  and  $g$  have an equaliser on each irreducible component and then take the union of these. So, without loss of generality, assume  $X$  is irreducible. Then, by Schur's lemma for crystals,  $f$  and  $g$  have to either agree or disagree entirely as maps of sets. Thus either  $X$  or  $\{0\}$  is an equaliser of  $f$  and  $g$  in  $\text{Crys}_{\mathfrak{g}}$ , and either  $\{0\} = F(\{0\})$  or  $FX$  is an equaliser of  $f$  and  $g$  in  $\text{Set}_{\bullet}$ .  $\square$

Thus we have the following.

**Theorem 4.3.** *The comonad  $\mathbb{U} = (U = F \circ G, \eta, \mu)$  gives an equivalence of categories  $J_{\mathbb{U}} : \text{Crys}_{\mathfrak{g}} \rightarrow \text{Set}_{\bullet, \mathbb{U}}$  between the category of  $\mathfrak{g}$  crystals and the category of algebras over the comonad  $U$ . Thus we have classified all crystals as coalgebras over the comonad  $U$ .*

**Remark** Explicitly, we see that

$$U = FG : A \mapsto \bigsqcup_{\alpha \in \Phi_+} \bigsqcup_{\substack{f \in \text{Hom}(FB(\alpha), A) \\ f \neq 0}} F(B(\alpha)_f)$$

with

$$\eta_{B(\alpha)} : B(\alpha) \rightarrow \bigsqcup_{\beta \in \Phi_+} \bigsqcup_{\substack{f \in \text{Hom}(FB(\beta), FB(\alpha)) \\ f \neq 0}} B(\beta)_f,$$

$$b \mapsto (b)_{\text{id}_{FB(\alpha)}} \in F(B(\alpha)_{\text{id}_{FB(\alpha)}})$$

and

$$\varepsilon_A : \bigsqcup_{\alpha \in \Phi_+} \bigsqcup_{\substack{f \in \text{Hom}(FB(\alpha), A) \\ f \neq 0}} F(B(\alpha)_f) \rightarrow A,$$

$$(b)_f \mapsto f(b)$$

so

$$\Delta_A : \bigsqcup_{\alpha \in \Phi_+} \bigsqcup_{\substack{f \in \text{Hom}(FB(\alpha), A) \\ f \neq 0}} F(B(\alpha)_f) \rightarrow \bigsqcup_{\beta \in \Phi_+} \bigsqcup_{\substack{g \in \text{Hom}(FB(\beta), FG(A)) \\ g \neq 0}} F(B(\beta)_g)$$

$$(b)_f \mapsto (b)_{x \mapsto (x)_f} \in F(B(\alpha)_{x \mapsto (x)_f})$$

where we have the maps  $B(\alpha) \rightarrow FG(A)$ ,  $x \mapsto (x)_f$ . For notational purposes, let us denote these maps  $s_f : x \mapsto (x)_f$ . From here we can explicitly see the coalgebra structure of each  $B(\alpha)$  over  $FG$  is given by a map

$$\zeta : F(B(\alpha)) \rightarrow FG(F(B(\alpha))), \quad b \mapsto (b)_{\text{id}_{F(B(\alpha))}}$$

which extends to the coalgebra structure of a general crystal  $X = \bigsqcup_{j \in J} B(\beta_j)$  as follows:

$$\zeta : F(X) \rightarrow FG(F(X)), \quad b \mapsto (b)_{(F(B(\beta_j)) \hookrightarrow FX)} \quad \text{for } b \in F(B(\beta_j)).$$

**4.3. Recovering the Crystal Structure.** Given a pointed set  $A$  with a coalgebra structure  $(A, \zeta_A)$  over our comonad  $U = FG$ , we know from the above that  $A$  carries a crystal structure that has been forgotten by the forgetful functor  $F$ . In fact, there is a way of recovering this crystal structure from the coalgebra structure. We regain the Kashiwara operator  $\tilde{f}_i$  (and similarly  $\tilde{e}_i$ ) via the following composition:

$$A \xrightarrow{\zeta_A} FG(A) \xrightarrow{\tilde{f}_i} FG(A) \xrightarrow{\varepsilon_A} A.$$

We also regain the weight function via

$$A \rightarrow FG(A) \rightarrow \Phi$$

where the last arrow is the map  $(b)_f \mapsto \text{wt}(b)$ .

**4.4. The link with  $\mathcal{B}$ .** For comonads  $C, C'$ , if we have a morphism of  $\psi : C \Rightarrow C'$  of comonads then we may give a  $C$ -coalgebra  $(M, \Delta_M)$  a  $C'$ -coalgebra structure via the composition of maps  $(\psi_M \otimes \text{id}) \circ \Delta_M : M \rightarrow C(M) \rightarrow C'(M)$ . Thus we may push forward  $C$ -coalgebras to  $C'$ -coalgebras via  $\psi$ .

We have already seen that objects of  $\text{Crys}_{\mathfrak{g}}$  have a  $\mathcal{B}$ -comodule structure, so we may ask whether there is a morphism between our comonad  $FG$  and our coalgebra  $\mathcal{B}$  that gives  $FG$ -coalgebras a  $\mathcal{B}$ -comodule structure as above. For this to make sense, we view  $\mathcal{B}$  as the functor  $H := \mathcal{B} \otimes - : \text{Set}_{\bullet} \rightarrow \text{Set}_{\bullet}$  on  $\text{Set}_{\bullet}$ . This comes with natural transformation acting as comultiplication on the comonad  $\Delta \otimes \text{id} : H \rightarrow H \circ H$ , however, as before, we do not have a counit map  $\varepsilon$ . It is clear that a pointed set  $A$  is a comodule over  $\mathcal{B}$  if and only if  $A$  is a coalgebra over the comonad  $H$ .

**Proposition 4.4.** *The  $\mathcal{B}$ -comodule structure on each  $B(\alpha)$ , and hence by extension every object in  $\text{Crys}_{\mathfrak{g}}$ , arises as a result of pushing forward the  $U$ -coalgebra structure via the natural transformation  $\theta : U \Rightarrow \mathcal{B} \otimes -$ ,*

$$\theta_A : \bigsqcup_{\alpha \in \Phi_+} \bigsqcup_{\substack{f \in \text{Hom}(FB(\alpha), A) \\ f \neq 0}} F(B(\alpha)_f) \rightarrow \bigsqcup_{\alpha \in \Phi_+} B(\alpha) \otimes B(\alpha)^* \otimes A$$

$$(b)_f \mapsto (b \otimes u_{-\alpha}) \otimes f(u_{\alpha})$$

for  $(b)_f \in F(B(\alpha)_f)$  indexed by  $f \in \text{Hom}(FB(\alpha), A)$ .

*Proof.* Firstly, for pointed sets  $A, A'$  with a morphism  $g : A \rightarrow A'$  between them,  $\theta_{A'} \circ FG(g)((b)_f) = \theta_{A'}((b)_{g \circ f}) = (b \otimes u_{-\alpha}) \otimes g \circ f(u_{\alpha})$  and  $(\text{id} \otimes g) \circ \theta_A((b)_{g \circ f}) = (\text{id} \otimes g)(b \otimes u_{-\alpha}) \otimes f(u_{\alpha}) = (b \otimes u_{-\alpha}) \otimes g \circ f(u_{\alpha})$ , where  $(b)_f \in F(B(\alpha)_f)$  is indexed by  $f \in \text{Hom}(FB(\alpha), A)$ . So  $\theta : FG \Rightarrow H$  defines a natural transformation. Thus we can give each  $B(\alpha)$ , and by extension

every object in  $\mathcal{C}r\mathcal{Y}\mathcal{S}_{\mathbf{g}}$ , the structure of a  $\mathcal{B}$ -comodule via  $\theta$ . It remains to check that this coaction agrees with the coaction we already have. Indeed,

$$F(B(\alpha)) \rightarrow FG(F(B(\alpha))), \quad b \mapsto (b)_{\text{id}_{F(B(\alpha))}}$$

induces

$$F(B(\alpha)) \rightarrow H(B(\alpha)) = \mathcal{B} \otimes B(\alpha), \quad b \mapsto (b \otimes u_{-\alpha}) \otimes u_{\alpha}.$$



## 5. THE STRUCTURE OF $U$

In this section we shall look at the generalisation of bialgebras to the setting of monadic functors through the study of monoidal functors. For more on these concepts see [15], [16], [1] and [14].

**5.1. Monoidal Functors.** In the setting of functors, the notion of a bimonad is not obvious. The subtlety comes from the lack of symmetry when composing functors - there is no natural twist  $A \circ B \Rightarrow B \circ A$  for functors  $A, B$  on a category  $\mathcal{C}$ . Recall that, for a coalgebra  $H$  (in, say, the category of vector spaces) the categories of modules and comodules of  $H$  inherit a monoidal structure. We wish to generalise the property of bialgebras that allows us to encode the monoidal structure of the categories of modules and comodules, as seen in the previous section. To generalise this, recall the definition of a monoidal functor.

**Definition** A comonadic functor  $T$  on a category  $\mathcal{C}$  is said to be *monoidal* (or a *bicomonad*) if there is a natural tranformation

$$\chi_{A,B} : T(A) \otimes T(B) \Rightarrow T(A \otimes B)$$

and a morphism  $\mathbb{I} \rightarrow T(\mathbb{I})$ , where  $\mathbb{I}$  is taken to be the identity of the tensor product, satisfying the following compatibility conditions for the monad structure:

$$\begin{array}{ccc}
T(A) \otimes T(B) & \xrightarrow{\chi_{A,B}} & T(A \otimes B) \\
\Delta_A \otimes \Delta_B \downarrow & & \Delta_{A \otimes B} \downarrow \\
TT(A) \otimes TT(B) & \xrightarrow{\chi_{T(A), T(B)}} T(T(A) \otimes T(B)) \xrightarrow{T(\chi_{A,B})} & TT(A \otimes B)
\end{array},$$

$$\begin{array}{ccccc}
T(A \otimes \mathbb{I}) & \leftarrow & T(A) \otimes T(\mathbb{I}) & \leftarrow & T(A) \otimes \mathbb{I} \\
& \searrow & & \swarrow & \\
& & T(A) & & \\
& \swarrow & & \searrow & \\
T(\mathbb{I} \otimes A) & \leftarrow & T(\mathbb{I}) \otimes T(A) & \leftarrow & \mathbb{I} \otimes T(A) \\
& \searrow & & \swarrow & \\
& & T(A) & &
\end{array},$$

Dually, we may define *opmonoidal* monadic functors (or *comonads*) with natural transformations  $\chi_{A,B} : T(A \otimes B) \Rightarrow T(A) \otimes T(B)$  and morphisms  $T(\mathbb{I}) \rightarrow \mathbb{I}$  satisfying analogous compatibility conditions.

**Remark** For a comonadic functor, the property of being monoidal gives a monoidal structure to the category of coalgebras. In fact, it has been proven in [15] that monoidal structures on  $\mathcal{D}_{\mathbb{U}}$  compatible with the forgetful functor correspond to monoidal structures on  $U$ . Because of this, we may think of the functor as bimonadic (or, perhaps more accurately, bicomonadic). The coaction on a tensor product of two coalgebras is given by the following composition:

$$A \otimes B \rightarrow T(A) \otimes T(B) \rightarrow T(A \otimes B)$$

where the first arrow is given by the respective coactions of  $A$  and  $B$ , and the second given by  $\chi$ . We see analogous results for the category of algebras over an opmonoidal monadic functor.

**Lemma 5.1** ([15]). *Suppose  $G : \mathcal{D} \rightarrow \mathcal{C}$  is a comonadic functor, and so  $\mathbb{U} = (U = FG, \Delta, \epsilon)$  is a monad with equivalence of categories  $J_{\mathbb{U}} : \mathcal{C} \rightarrow \mathcal{D}_{\mathbb{U}}$ . Suppose further that the forgetful functor  $F$  is a strong monoidal functor, with isomorphism  $\kappa_F : F(\bullet \otimes \bullet) \xrightarrow{\cong} F(\bullet) \otimes F(\bullet)$ . Then the image of  $\epsilon_A \otimes \epsilon_B$  under  $\text{Hom}(FG(A) \otimes FG(B), A \otimes B) \cong \text{Hom}(F(GA \otimes GB), A \otimes B) \cong \text{Hom}(GA \otimes GB, G(A \otimes B))$  induces a monoidal structure on  $U$  with  $\chi$  defined by  $FG(A \otimes B) \rightarrow F(GA \otimes GB) \xrightarrow{\cong} FGA \otimes FGB$ .*

From this, we get an extension of the Barr-Beck theorem.

**Theorem 5.2.** *Let  $\mathcal{C}, \mathcal{D}$  be monoidal categories. Suppose  $F : \mathcal{C} \rightarrow \mathcal{D}$  has a right adjoint  $G : \mathcal{D} \rightarrow \mathcal{C}$ . Then  $J_{\mathbb{U}} : \mathcal{C} \rightarrow \mathcal{D}_{\mathbb{U}}$  is an equivalence of monoidal categories if and only if:*

- i)  $F$  is a strong monoidal functor;
- ii)  $F$  reflects isomorphisms;
- iii) If a pair  $f, g : A \rightarrow B$  are morphisms in  $\mathcal{C}$  such that  $F(f), F(g)$  have a split equaliser  $h : H \rightarrow F(A)$  in  $\mathcal{D}$  then  $f, g$  have an equaliser  $e : E \rightarrow A$  in  $\mathcal{C}$  such that  $F(e) = h, F(E) = H$ ;

**Remark** In the context of our category of pointed sets,  $\text{Set}_{\bullet}, \mathbb{I}$  is just the pointed singleton set  $\{*\}_{\bullet}$ . It is clear that the forgetful functor  $F$  essentially preserves tensor products, giving us the following.

**Proposition 5.3.** *The monoidal structure of  $\text{Crys}_{\mathfrak{g}}$  given by the tensor product of crystals can be seen as a result of the monoidal structure of  $U$ .*

**Remark** Our comonadic functor  $U = FG$  has a monoidal structure as follows:

$$\tilde{\chi}_{A,B} : U(A) \otimes U(B) = \bigsqcup_{\substack{\alpha \in \Phi_+ \\ \beta \in \Phi_+}} \bigsqcup_{\substack{f : FB(\alpha) \rightarrow A \\ g : FB(\beta) \rightarrow B}} FB(\alpha)_f \otimes FB(\beta)_g$$

$$\begin{aligned} &\longrightarrow U(A \otimes B) = \bigsqcup_{\gamma \in \Phi_+} \bigsqcup_{h: FB(\gamma) \rightarrow A \times B} FB(\gamma)_h, \\ b_f \otimes b'_g &\mapsto (\vartheta_{\alpha, \beta}(b \otimes b'))_{(f \otimes g) \circ \vartheta_{\alpha, \beta}^{-1}} \end{aligned}$$

with

$$\mathbb{I} \rightarrow U(\mathbb{I}) = \bigsqcup_{\alpha \in \Phi_+} \bigsqcup_{f: FB(\alpha) \rightarrow \{*\}} FB(\gamma)_h, \quad * \mapsto (v_0)_{(B(0) \rightarrow \{*\})}$$

where  $\vartheta_{\alpha, \beta} : B(\alpha) \otimes B(\beta) \rightarrow \bigsqcup_{\gamma \in \Gamma_{\alpha, \beta}} B(\gamma)$  is the decomposition of  $B(\alpha) \otimes B(\beta)$  into irreducible crystals, and  $f \otimes g$  is restricted to the connected component containing  $b$ .

### Part 3. Linearised Crystal Bases and a Quantum Group at $q = 0$

#### 6. LINEARISED CRYSTALS AND A CRYSTAL BIALGEBRA

**Definition** Consider the category  $\mathcal{C}$  whose objects are free abelian groups on crystal bases arising from integrable  $U_q(\mathfrak{g})$ -representations, and whose morphisms are just those of abelian groups. For crystals  $B(\alpha)$ ,  $\alpha \in \Phi_+$ , we will denote by  $\mathbb{B}(\alpha)$  the free abelian group  $\mathbb{Z}B(\alpha)$ . Then we have maps

$$\iota : \mathbb{B}(0) \cong \mathbb{Z} \rightarrow \mathbb{B}(-\alpha) \otimes \mathbb{B}(\alpha), \quad 1 \mapsto \sum_{b \in B(\alpha)} b^\vee \otimes b,$$

$$\epsilon : \mathbb{B}(\alpha) \otimes \mathbb{B}(-\alpha) \rightarrow \mathbb{Z} \cong \mathbb{B}(0), \quad b \otimes b' \mapsto \delta_{b^\vee, b'},$$

the *coevaluation* and *evaluation* respectively, and similarly  $\iota' : \mathbb{B}(0) \rightarrow \mathbb{B}(\alpha) \otimes \mathbb{B}(-\alpha)$ ,  $\epsilon' : \mathbb{B}(-\alpha) \otimes \mathbb{B}(\alpha)$ .

**Proposition 6.1.** *The compositions*

$$\mathbb{B}(\alpha) \cong \mathbb{B}(\alpha) \otimes \mathbb{B}(0) \xrightarrow{id \otimes \iota} \mathbb{B}(\alpha) \otimes \mathbb{B}(-\alpha) \otimes \mathbb{B}(\alpha) \xrightarrow{\epsilon \otimes id} \mathbb{B}(\alpha)$$

$$\mathbb{B}(\alpha) \cong \mathbb{B}(0) \otimes \mathbb{B}(\alpha) \xrightarrow{\iota' \otimes id} \mathbb{B}(\alpha) \otimes \mathbb{B}(-\alpha) \otimes \mathbb{B}(\alpha) \xrightarrow{id \otimes \epsilon'} \mathbb{B}(\alpha)$$

*both agree with the identity. Thus  $\mathcal{C}$  is rigid monoidal.*

**Remark** Let us denote by  $\tau$  the usual twist of abelian groups  $\tau : A \otimes B \rightarrow B \otimes A$ ,  $a \otimes b \mapsto b \otimes a$ . This makes our category braided monoidal. As discussed previously, this braiding is preferable to a  $\mathbb{Z}$ -linear extension of Henriques and Kamnitzer's commutator of crystals. This commutator of crystals does not satisfy the usual hexagon axiom, and so would not have given us a monoidal structure but instead we would have a *cactus* (or *coboundary*) category (see [5]).

**Remark** In Section 2, we defined a crystal  $\mathcal{B} = \bigsqcup_{\alpha \in \Phi_+} B(\alpha) \otimes B(-\alpha)$  which exhibited the structure of a noncounital bialgebra whose comodules included (but were not limited to) the crystals arising from  $U_q(\mathfrak{g})$ -modules. Let us denote by  $\mathbb{B}$  the corresponding object in our category  $\mathcal{C}$ ,  $\mathbb{B} = \mathbb{Z}\mathcal{B} = \bigoplus_{\alpha \in \Phi} \mathbb{B}(\alpha) \otimes \mathbb{B}(-\alpha)$ .



**Proposition 6.2.** *The object  $\mathbb{B}$  forms a bialgebra in  $\mathcal{C}$ . The multiplication being defined by extending the composition*

$$\begin{aligned}\mu : \mathbb{B}(\alpha) \otimes \mathbb{B}(-\alpha) \otimes \mathbb{B}(\beta) \otimes \mathbb{B}(-\beta) &\rightarrow \mathbb{B}(\alpha) \otimes \mathbb{B}(\beta) \otimes \mathbb{B}(-\beta) \otimes \mathbb{B}(-\alpha) \\ &\rightarrow (\oplus_{\gamma} \mathbb{B}(\gamma)) \otimes (\oplus_{\gamma'} \mathbb{B}(-\gamma)) \\ &\rightarrow \oplus_{\gamma} \mathbb{B}(\gamma) \otimes \mathbb{B}(-\gamma)\end{aligned}$$

where the first morphism uses our braiding  $\tau$  and the second comes from the decomposition of the tensor product of crystals into irreducible components. Each irreducible factor  $B(\gamma)$  of  $B(\alpha) \otimes B(\beta)$  corresponds to a unique irreducible factor  $B(-\gamma)$  of  $B(-\beta) \otimes B(-\alpha)$ , and the third morphism in the above composition projects onto each tensored pair  $\mathbb{B}(\gamma) \otimes \mathbb{B}(-\gamma)$  of these. The unit is  $\eta : \mathbb{Z} \xrightarrow{\sim} \mathbb{B}(0) \otimes \mathbb{B}(-0)$ . The comultiplication is defined by extending

$$\Delta : \mathbb{B}(\alpha) \otimes \mathbb{B}(-\alpha) \cong \mathbb{B}(\alpha) \otimes \mathbb{Z} \otimes \mathbb{B}(-\alpha) \xrightarrow{id \otimes \iota \otimes id} \mathbb{B}(\alpha) \otimes \mathbb{B}(-\alpha) \otimes \mathbb{B}(\alpha) \otimes \mathbb{B}(-\alpha)$$

with counit  $\epsilon$  as defined above.

*Proof.* The coassociativity and counit diagrams for  $\Delta$  and  $\epsilon$  are straightforward to check, as is checking that  $\eta$  is indeed a unit. The associativity of  $\mu$  follows from the properties of  $\tau$  and the associativity of the tensor product. It remains to verify that the multiplication and comultiplication interact to give a bialgebra. For  $b \in B(\alpha)$ ,  $d \in B(\beta)$  let us denote by  $b \cdot d$  the image of the element  $b \otimes d$  under the decomposition of the tensor product  $B(\alpha) \otimes B(\beta) \cong \sqcup_{\gamma} B(\gamma)$ . Then the product of  $b \otimes b' \in B(\alpha) \otimes B(-\alpha)$  and  $d \otimes d' \in B(\beta) \otimes B(-\beta)$  under  $\mu$  is  $(b \cdot d) \otimes (d' \cdot b')$  if  $(b \cdot d)$  and  $(d' \cdot b')^{\vee}$  lie in the same irreducible crystals in the decomposition of  $B(\alpha) \otimes B(\beta)$ , and is 0 otherwise. If  $(b \cdot d)$  and  $(d' \cdot b')^{\vee}$  lie in different irreducible crystals then  $\Delta \circ \mu((b \otimes b') \otimes (d \otimes d')) = 0$ . Also,  $\mu_{\mathbb{B} \otimes \mathbb{B}} \circ (\Delta \otimes \Delta)((b \otimes b') \otimes (d \otimes d')) = \sum_{b'' \in B(\alpha)} \sum_{d'' \in B(\beta)} \mu(b \otimes b''^{\vee} \otimes d \otimes d''^{\vee}) \otimes \mu(b'' \otimes b' \otimes d'' \otimes d')$ , the nonzero terms of which only occur when both  $b \cdot d$  and  $(d''^{\vee} \cdot b''^{\vee})^{\vee} = b'' \cdot d''$  lie in the same component, and  $b'' \cdot d''$  and  $(d' \cdot b')^{\vee}$  lie in the same component. Since this never occurs, this must also be zero. It remains to check when  $(b \cdot d)$  and  $(d' \cdot b')^{\vee}$  do lie in the same irreducible component,  $B(\gamma)$  say. In this case we have

$$(b \otimes b') \otimes (d \otimes d') \xrightarrow{\mu} (b \cdot d) \otimes (d' \cdot b') \xrightarrow{\Delta} \sum_{c \in B(\gamma)} ((b \cdot d) \otimes c^{\vee}) \otimes (c \otimes (d' \cdot b'))$$

whilst

$$(b \otimes b') \otimes (d \otimes d') \xrightarrow{\Delta \otimes \Delta} \sum_{b'' \in B(\alpha)} \sum_{d'' \in B(\beta)} b \otimes b''^{\vee} \otimes b'' \otimes d \otimes d''^{\vee} \otimes d'' \otimes d'$$

$$\xrightarrow{\mu_{\mathbb{B} \otimes \mathbb{B}}} \sum_{b'' \in B(\alpha)} \sum_{d'' \in B(\beta)} \mu(b \otimes b''^{\vee} \otimes d \otimes d''^{\vee}) \otimes \mu(b'' \otimes b' \otimes d'' \otimes d')$$

$$\begin{aligned}
&= \sum_{\substack{b'' \in B(\alpha) \\ d'' \in B(\beta) \\ b'' \cdot d'' \in B(\gamma)}} ((b \cdot d) \otimes (b'' \cdot d'')^\vee \otimes (b'' \cdot d'') \otimes (d' \cdot b')) \\
&= \sum_{c \in B(\gamma)} (b \cdot d) \otimes c^\vee \otimes c \otimes (d' \cdot b').
\end{aligned}$$

So  $\Delta$  is an algebra homomorphism. Similarly, if we say  $b \cdot d$  and  $d' \cdot b'$  lie in the same component,

$$\begin{aligned}
\epsilon((b \otimes b') \cdot (d \otimes d')) &= \epsilon(b \cdot d \otimes d' \cdot b') = \delta_{(b \cdot d)^\vee, d' \cdot b'} \\
&= \delta_{d^\vee \cdot b^\vee, d' \cdot b'} = \delta_{d^\vee, d'} \delta_{b^\vee, b'} \\
&= \epsilon(b \otimes b') \epsilon(d \otimes d').
\end{aligned}$$

since  $d^\vee \cdot b^\vee = d' \cdot b'$  if and only if  $d^\vee = d'$  and  $b^\vee = b'$ . The case when they do not lie in the same component is trivial, hence  $\epsilon$  is an algebra homomorphism too. Thus we have our result.  $\square$

**Remark** Immediately we see that the structure of  $\mathbb{B}$  is more well behaved than that of  $\mathcal{B}$ . This is mainly due to the rigidity of  $\mathcal{C}$  that the category of crystals was lacking as we saw previously, since the duality between  $\mathbb{B}(\alpha)$  and  $\mathbb{B}(-\alpha)$  plays a large role in defining the coalgebra structure.

**Proposition 6.3.** *Let  $\mathbb{B}_\lambda = \text{Span}_{\mathbb{Z}}\{b \otimes b' \in \mathcal{B} \mid \text{wt}(b) + \text{wt}(b') = \lambda\}$  for  $\lambda \in \Phi$ . Then  $\mathbb{B} = \bigoplus_{\lambda \in \Phi} \mathbb{B}_\lambda$  with  $\mathbb{B}_\lambda \cdot \mathbb{B}_{\lambda'} \subset \mathbb{B}_{\lambda+\lambda'}$  and  $\Delta(\mathbb{B}_\lambda) \subset \bigoplus_{\lambda=\lambda'+\lambda''} \mathbb{B}_{\lambda'} \otimes \mathbb{B}_{\lambda''}$ . That is,  $\mathbb{B}$  is a graded bialgebra.*

**Proposition 6.4.** *If we take a basis of  $\Phi$  of fundamental weights  $\{\Lambda_i \mid i \in I\}$  then  $\mathbb{B}$  is generated as an algebra by the  $B(\Lambda_i) \otimes B(-\Lambda_i)$  for  $i \in I$ .*

*Proof.* For each  $\sum_i n_i \Lambda_i \in \Phi_+$  the surjection

$$B(\Lambda_1)^{\otimes n_1} \otimes \dots \otimes B(\Lambda_k)^{\otimes n_k} \rightarrow B\left(\sum_i n_i \Lambda_i\right)$$

gives a surjection

$$\bigotimes_{i=1}^k (B(\Lambda_i) \otimes B(-\Lambda_i))^{\otimes n_i} \rightarrow B\left(\sum_i n_i \Lambda_i\right) \otimes B\left(-\sum_i n_i \Lambda_i\right)$$

onto a basis of  $\mathbb{B}(\sum_i n_i \Lambda_i) \otimes \mathbb{B}(-\sum_i n_i \Lambda_i)$ .  $\square$

**Remark** In the case of  $\mathfrak{sl}_2$ , the fundamental weight is  $1 \in \mathbb{N}$ , and  $B(1)$  has crystal graph  $y \rightarrow x$ . So we have four generators in  $B(1) \otimes B(-1)$ , namely

$$\begin{aligned}
a &= x \otimes x^\vee, & b &= y \otimes x^\vee, \\
c &= x \otimes y^\vee, & d &= y \otimes y^\vee,
\end{aligned}$$

retaining the notation from Part 1. Thus  $\mathbb{B}$  is a quotient of the free algebra  $\mathbb{Z}\langle a, b, c, d \rangle$ , viewed as a bialgebra via the comultiplication

$$\begin{aligned}
\Delta(a) &= a \otimes a + b \otimes c, & \Delta(b) &= a \otimes b + b \otimes d, \\
\Delta(c) &= c \otimes a + d \otimes c, & \Delta(d) &= c \otimes b + d \otimes d.
\end{aligned}$$

It is a straightforward (and perhaps tedious) calculation to verify the following.

**Proposition 6.5.** *In the case of  $\mathfrak{sl}_2$ ,  $x^i y^j \otimes (x^r y^s)^\vee = a^r c^{i-r} d^j$  if  $i \geq r$  ( $\Leftrightarrow j \leq s$ ) and  $x^i y^j \otimes (x^r y^s)^\vee = a^i b^{r-i} d^s$  if  $i \leq r$  ( $\Leftrightarrow j \geq s$ ) and multiplication is entirely determined by the relations*

$$cb = bc = db = dc = ba = ca = 0, \quad da = 1$$

in  $\mathbb{B}$ .

These relations in  $\mathbb{B}$  are closely related to those of the quantum coordinate ring, which we shall discuss at the very end of this section.

### 6.1. The Comodules of $\mathbb{B}$ .

**Remark** For each  $\alpha \in \Phi_+$  we can give  $\mathbb{B}(\alpha)$  a  $\mathbb{B}$ -comodule structure via the following map:

$$\begin{aligned} \mathbb{B}(\alpha) &\cong \mathbb{B}(\alpha) \otimes \mathbb{Z} \rightarrow \mathbb{B}(\alpha) \otimes \mathbb{B}(-\alpha) \otimes \mathbb{B}(\alpha) \hookrightarrow \mathbb{B} \otimes \mathbb{B}(\alpha) \\ b &\mapsto \sum_{b' \in B(\alpha)} b \otimes b'^\vee \otimes b'. \end{aligned}$$

Hence any object of  $\mathcal{C}$  is a  $\mathbb{B}$ -comodule (in the category of free abelian groups). In fact, these are essentially all of the  $\mathbb{B}$ -comodules.

**Theorem 6.6.** *Any  $\mathbb{B}$ -comodule is isomorphic to an abelian group of the form  $\mathbb{Z}X$  where  $X$  is a crystal.*

*Proof.* Let  $M$ , a free abelian group, be a  $\mathbb{B}$ -comodule. Then there are  $\mathbb{Z}$ -linear morphisms  $A_{b,b'}^\alpha : M \rightarrow M$  indexed by  $\alpha \in \Phi_+$ ,  $b, b' \in B(\alpha)$ , such that  $\Delta_M(m) = \sum_{\alpha \in \Phi_+} \sum_{b, b' \in B(\alpha)} b \otimes b'^\vee \otimes A_{b,b'}^\alpha(m)$  for all  $m \in M$ . Then, by assumption, we have

$$\begin{aligned} &\sum_{\alpha \in \Phi_+} \sum_{b, b' \in B(\alpha)} \sum_{\beta \in \Phi} \sum_{d, d' \in B(\beta)} b \otimes b'^\vee \otimes d \otimes d'^\vee \otimes A_{d,d'}^\beta A_{b,b'}^\alpha(m) \\ &= \sum_{\alpha \in \Phi_+} \sum_{b, b' \in B(\alpha)} \sum_{d \in B(\alpha)} b \otimes d \otimes d^\vee \otimes b'^\vee \otimes A_{b,b'}^\alpha(m) \\ &= \sum_{\alpha \in \Phi_+} \sum_{b \in B(\alpha)} A_{b,b}^\alpha(m) \end{aligned}$$

for all  $m \in M$ . That is,

$$A_{d,d'}^\beta A_{b,b'}^\alpha = \delta_{\alpha,\beta} \delta_{b',d} A_{b,d}^\alpha, \quad \sum_{\alpha \in \Phi_+} \sum_{b \in B(\alpha)} A_{b,b}^\alpha = \text{Id}_M.$$

From this we see that  $A_{b,b}^\alpha$  form perpendicular idempotents on  $M$ , and hence  $M = \oplus_\alpha \oplus_b M_b^\alpha$  where  $M_b^\alpha = A_{b,b}^\alpha M$ . Let us denote  $M^\alpha = \oplus_{b \in B(\alpha)} M_b^\alpha$ . Then, for any  $m \in M_b^\alpha$ ,  $m = A_{b,b}^\alpha m$  and so

$$\Delta_M(m) = \sum_{\beta \in \Phi_+} \sum_{d, d' \in B(\beta)} d \otimes d'^\vee \otimes A_{d,d'}^\beta A_{b,b}^\alpha(m) = \sum_{b' \in B(\alpha)} b \otimes b'^\vee \otimes A_{b,b'}^\alpha(m).$$

Note also that  $A_{b',b}^\alpha A_{b,b'}^\alpha = A_{b,b}^\alpha$ , so when restricted to  $M^\alpha$  we see that these give isomorphisms of free abelian groups between each pair  $M_b^\alpha, M_{b'}^\alpha$ . It then follows from picking a free basis for some  $M_b^\alpha$ , and using  $A_{b,b'}^\alpha$  to then obtain free bases of  $M_{b'}^\alpha$ , that  $M^\alpha \cong \mathbb{B}(\alpha)^{\oplus r}$  where  $r$  is the rank of each  $M_b^\alpha$ . From this we see that  $M$  is of the form  $\mathbb{Z}X$  for some crystal  $X$ .  $\square$

**Remark** As a comodule,  $\mathbb{B} \cong \bigoplus_\alpha \bigoplus_{b' \in B(-\alpha)} \mathbb{B}(\alpha)$  via  $b \otimes b' \mapsto (b)_{b'}$  in the copy of  $\mathbb{B}(\alpha)$  indexed by  $b' \in B(-\alpha)$ . Under this isomorphism, multiplication becomes  $(b)_{b'} \cdot (d)_{d'} = (b \cdot d)_{d' \cdot b'}$  whenever this is well defined, and 0 otherwise, and comultiplication becomes  $(b)_{b'} \mapsto \sum_{b'' \in B(\alpha)} (b)_{b' \vee} \otimes (b'')_{b'}$ .

**Remark** Note that the isomorphism in Theorem 7.1 above is not canonical, and depends on choosing appropriate bases for each  $M^\alpha$ . There is, however, a way to get round this.

**Definition** A *based  $\mathbb{B}$ -comodule* is a pair  $(M, X)$  such that  $M$  is a  $\mathbb{B}$ -comodule and  $X$  is a free basis of  $M$  such that  $X = \bigsqcup_\alpha \bigsqcup_b X_b^\alpha$  where  $X_b^\alpha = X \cap M_b^\alpha$ , and each  $A_{b,b'}^\alpha$  restricts to a bijection between the sets  $X_b^\alpha \rightarrow X_{b'}^\alpha$ . This is equivalent to having chosen a basis  $X_{b_\alpha}^\alpha$  for each  $M_{b_\alpha}^\alpha$  where  $b_\alpha$  is the highest weight element of  $B(\alpha)$  for  $\alpha \in \Phi_+$ . A morphism of based comodules  $(M, X) \rightarrow (N, Y)$  is a morphism of comodules  $f : M \rightarrow N$  such that  $f(X) \subset Y$ . Since such a morphism commutes with the comultiplication it also commutes with the  $A_{b,b'}^\alpha$ . The direct sum of two based comodules is  $(M, X) \oplus (N, Y) = (M \oplus N, X \sqcup Y)$  and their tensor product is  $(M, X) \otimes (N, Y) = (M \otimes N, X \otimes Y = \{x \otimes y \mid x \in X, y \in Y\})$ . Here we are making use of the fact that comodules over a bialgebra form a monoidal category, so the category of based comodules is also monoidal.

**Theorem 6.7.** *The functor  $X \mapsto (\mathbb{Z}X, X)$  gives an equivalence of categories between the category of crystals and the category of based  $\mathbb{B}$ -comodules.*

*Proof.* It is clear that  $X \mapsto (\mathbb{Z}X, X)$  is functorial. We shall construct a quasi-inverse  $H$  as follows. Let the underlying pointed set of  $H(M, X)$  be that of  $X \sqcup \{0\}$ . For  $x \in X_b^\alpha$  we set  $\text{wt}(x) = \text{wt}(b)$ ,  $\phi_i(x) = \phi_i(b)$ ,  $\epsilon_i(x) = \epsilon_i(b)$ ,  $\tilde{e}_i x = A_{b, \tilde{e}_i b}^\alpha x$  and  $\tilde{f}_i x = A_{b, \tilde{f}_i b}^\alpha x$ . It is then clear that this defines a crystal structure on  $H(M, X)$ , and that  $H$  is a quasi-inverse to  $F$ . Here, the image under  $H$  of a morphism of based comodules is the restriction to the underlying set, which commutes with the crystal structure given above.  $\square$

As mentioned above, since  $\mathbb{B}$  is a bialgebra, its comodules (and hence based comodules) form a monoidal category.

**Proposition 6.8.** *The functor above gives an equivalence of monoidal categories.*

*Proof.* For  $\alpha, \beta \in \Phi_+$ , the comodule structure of  $\mathbb{B}(\alpha) \otimes \mathbb{B}(\beta)$  is

$$\Delta(b \otimes d) = \sum (b \cdot d) \otimes (d'^\vee \cdot b'^\vee) \otimes (b' \cdot d') = \sum (b \cdot d) \otimes (b' \cdot d')^\vee \otimes (b' \cdot d')$$

where both summations are taken over all  $b' \in B(\alpha)$  and  $d' \in B(\beta)$  such that  $b' \cdot d'$  and  $(d'^\vee \cdot b'^\vee)^\vee = b' \cdot d'$  lie in the same connected component. Since all terms of this connected component appear as some product  $b' \cdot d'$ , we can then rewrite this as  $\sum_{c \in B(\gamma)} (b \cdot d) \otimes c^\vee \otimes c$ . This is the same comultiplication of  $b \otimes d$  as when viewed as an element of  $\mathbb{Z}(B(\alpha) \otimes B(\beta))$  under its decomposition into irreducible components. Our result then follows.  $\square$

We can reformulate the definition of based comodule in terms of the basis being a  $\mathcal{B}$ -comodule as follows.

**Proposition 6.9.** *A pair  $(M, X)$  is a based comodule, where  $X$  is a free basis of the abelian group  $M$ , if and only if  $M$  is a  $\mathbb{B}$ -comodule,  $X$  is a  $\mathcal{B}$ -comodule, and these structures are compatible in the following sense:  $X$  decomposes as  $X = \bigsqcup_\alpha \bigsqcup_b X_b^\alpha$  where  $X_b^\alpha = X \cap M_b^\alpha$  for the decomposition of  $M$  discussed above, and if  $\Delta_{\mathbb{B}}(x) = \sum_\alpha \sum_{b, b'} b \otimes b'^\vee \otimes A_{b, b'}^\alpha(x)$  and  $\Delta_{\mathcal{B}}(x) = b \otimes b'^\vee \otimes x'$ , for  $x \in X$ , then  $A_{b, b'}^\alpha(x) = x'$  and  $x \in X_b^\alpha$ .*

*Proof.* Suppose we have a pair  $(M, X)$  as above. Without loss of generality, we may assume that  $M = M^\alpha$ , so  $X = X^\alpha$ . We have already seen in Part 1 that, for  $x \in X$ ,  $\Delta_{\mathcal{B}}(x) = b \otimes b_\alpha^\vee \otimes y$  for some  $b \in B(\alpha)$ ,  $y \in X$ , and  $\Delta_{\mathcal{B}}(y) = b_\alpha \otimes b_\alpha^\vee \otimes y$ . Then  $x \in X_b^\alpha$  and  $A_{b, b_\alpha}^\alpha(x) = y \in X_{b_\alpha}^\alpha$ . Thus  $A_{b, b_\alpha}^\alpha$  maps  $X_b$  to  $X_{b_\alpha}$ . Now let  $y \in X_{b_\alpha}^\alpha$  and let  $b \in B(\alpha)$ . Then  $A_{b_\alpha, b}^\alpha(y) \in X_b^\alpha$  and so can be written as a sum  $\sum_{x \in X_b^\alpha} a_x x$  for  $a_x \in \mathbb{Z}$ . Then  $y = A_{b_\alpha, b_\alpha}^\alpha(y) = A_{b, b_\alpha}^\alpha A_{b_\alpha, b}^\alpha(y) = \sum_{x \in X_b^\alpha} a_x A_{b, b_\alpha}^\alpha(x)$ . Since  $X_{b_\alpha}^\alpha$  is a free basis for  $M_{b_\alpha}^\alpha$  and  $A_{b, b_\alpha}^\alpha$  restricts to an isomorphism  $M_b^\alpha \rightarrow M_{b_\alpha}^\alpha$  that sends  $X_b^\alpha$  to  $X_{b_\alpha}^\alpha$ , we have that only one  $a_x \neq 0$ , in which case it equals 1. It follows then that  $(M, X)$  is a based comodule. For the converse, we have seen that a based comodule  $(M, X)$  gives rise to a crystal structure on  $X$ , and hence the structure of a  $\mathcal{B}$ -comodule. The compatibility then follows from construction.  $\square$

**Remark** Although we can exhibit duality in the category  $\mathcal{C}$ , this does not give us duality in the category of comodules - the (co)evaluation maps are not comodule morphisms. As we have seen in Part 1, this corresponds to a lack of duality in the category of crystals. Therefore  $\mathbb{B}$  does not have an antipode. There is, however, a bialgebra antimorphism  $S : \mathbb{B} \rightarrow \mathbb{B}^{\text{op}, \text{cop}}$  defined by  $b \otimes b' \mapsto b' \otimes b \in B(-\alpha) \otimes B(\alpha) = B(w_0\alpha) \otimes B(-w_0\alpha)$  for  $b \otimes b' \in B(\alpha) \otimes B(-\alpha)$ . Here  $w_0$  is the longest element of the Weyl group.

**Proposition 6.10.** *The antihomomorphism  $S$  connects the coalgebra structure of (based) comodules  $\mathbb{B}(\alpha)$  and  $\mathbb{B}(-\alpha)$  via the commutative diagram*

$$\begin{array}{ccc}
\mathbb{B} \otimes \mathbb{B}(\alpha) \otimes \mathbb{B}(-\alpha) & \xleftarrow{\Delta_{\mathbb{B}(\alpha)} \otimes id} \mathbb{B}(\alpha) \otimes \mathbb{B}(-\alpha) & \xrightarrow{id \otimes \Delta_{\mathbb{B}(-\alpha)}} \mathbb{B}(\alpha) \otimes \mathbb{B} \otimes \mathbb{B}(-\alpha) \\
\downarrow id \otimes \epsilon & & \downarrow (id \otimes \tau) \circ (\epsilon \otimes id) \\
\mathbb{B} & \xrightarrow{S} & \mathbb{B}
\end{array}$$

*Proof.* This is just a straightforward verification on the basis.  $\square$

**Remark** Note that, since  $\mathbb{B}$  and  $\mathbb{B}'$  share the same coalgebra structure, all of the above results hold for  $\mathbb{B}'$  as well.

## 6.2. Relation to The Crystal Functor.

**Remark** Recall from Part 2 that the functor

$$U : \text{Set}_\bullet \rightarrow \text{Set}_\bullet, X \mapsto \bigsqcup_{\alpha \in \Phi_+} \bigsqcup_{\substack{f: FB(\alpha) \rightarrow X \\ f \neq 0}} FB(\alpha)$$

is comonadic, with its category of coalgebras equivalent to the category of crystals,  $\text{Crys}_\mathfrak{g}$ . If, for pointed sets  $A, B$ , we consider  $\text{Hom}_{\text{Set}_\bullet}(A, B)$  as a pointed set

$$\underline{\text{Hom}}_{\text{Set}_\bullet}(A, B) = \{f : A \rightarrow B \mid f \neq 0\} \sqcup \{0 : A \rightarrow B\}$$

then this functor can be rewritten  $X \mapsto \bigsqcup_{\alpha \in \Phi_+} FB(\alpha) \otimes \underline{\text{Hom}}_{\text{Set}_\bullet}(FB(\alpha), A)$ . Note that this functor clearly does not preserve coproducts.

For a free abelian group  $A$ , we have

$$\mathbb{B} \otimes A = \bigoplus_{\alpha} \mathbb{B}(\alpha) \otimes \mathbb{B}(\alpha)^\vee \otimes A \cong \bigoplus_{\alpha} \mathbb{B}(\alpha) \otimes \underline{\text{Hom}}_{\mathbb{Z}}(\mathbb{B}(\alpha), A)$$

given by  $b \otimes b' \otimes a \mapsto b \otimes [x \mapsto \epsilon(x \otimes b')a]$ . Thus the functor  $A \mapsto \mathbb{B} \otimes A$  is remarkably similar to the crystal functor  $U$ . The coalgebra structure can be encoded as a comonadic structure on the functor  $V : A \mapsto \bigoplus_{\alpha} \mathbb{B}(\alpha) \otimes \underline{\text{Hom}}_{\mathbb{Z}}(\mathbb{B}(\alpha), A)$ ,  $\Delta : V \Rightarrow VV$ , given by

$$\bigoplus_{\alpha} \mathbb{B}(\alpha) \otimes \underline{\text{Hom}}_{\mathbb{Z}}(\mathbb{B}(\alpha), A) \rightarrow \bigoplus_{\alpha} \mathbb{B}(\alpha) \otimes \underline{\text{Hom}}_{\mathbb{Z}}(\mathbb{B}(\alpha), VA)$$

$$b \otimes f \mapsto b \otimes f^\sim$$

where  $f^\sim(b') = b' \otimes f$ . Again, note the similarity between this and the comultiplication on the crystal functor  $U$  given by  $\Delta : U \Rightarrow UU$ ,  $\Delta_A : \bigsqcup_{\alpha \in \Phi_+} \bigsqcup_{\substack{f: B(\alpha) \rightarrow X \\ f \neq 0}} B(\alpha) \rightarrow \bigsqcup_{\alpha \in \Phi_+} \bigsqcup_{\substack{f: B(\alpha) \rightarrow UA \\ f \neq 0}} B(\alpha)$ ,  $(b)_f \mapsto (b)_{x \mapsto x_f}$ . Under the identification of  $U$  with the functor  $X \mapsto \bigsqcup_{\alpha \in \Phi_+} B(\alpha) \otimes \underline{\text{Hom}}_{\text{Set}_\bullet}(B(\alpha), A)$  this becomes  $b \otimes f \mapsto b \otimes f^\sim$  where again  $f^\sim(b') = b' \otimes f$ .

Therefore we can view the functor  $V$  as a kind of linear extension of  $U$ . It is essentially due to the fact that the functors  $\underline{\text{Hom}}_{\mathbb{Z}}(\mathbb{B}(\alpha), -)$  preserve coproducts that we obtain a bialgebra  $\mathbb{B}$  out of the functor  $V$ , and the fact that  $\underline{\text{Hom}}_{\text{Set}_\bullet}(B(\alpha), -)$  does not means that we do not obtain a bialgebra out of  $U$ .

### 6.3. Duality.

**Remark** Since  $\mathbb{B}$  is a direct sum of the finite rank free abelian groups with respective bases  $B(\alpha) \otimes B(-\alpha)$  ranging over  $\alpha \in \Phi_+$  the dual to  $\mathbb{B}$  is isomorphic to the product  $\prod_{\alpha \in \Phi_+} \mathbb{B}(\alpha) \otimes \mathbb{B}(-\alpha)$ . We shall denote elements of this dual by formal sums  $\sum_{b, b'} a_{b, b'} \widehat{b \otimes b'}$  ranging over all  $b \otimes b' \in B(\alpha) \otimes B(-\alpha)$  and  $\alpha \in \Phi_+$ , where  $\sum_{b, b'} a_{b, b'} \widehat{b \otimes b'}(d \otimes d') = a_{d, d'}$  for  $d \otimes d' \in B(\beta) \otimes B(-\beta)$  for some  $\beta \in \Phi_+$ . Then the bialgebra structure on this dual  $\mathbb{B}^\vee$ , has

$$\left( \sum_{b, b'} a_{b, b'} \widehat{b \otimes b'} \right) \cdot \left( \sum_{b, b'} a'_{b, b'} \widehat{b \otimes b'} \right) = \sum_{b, b'} \sum_d a_{b, d^\vee} a'_{d, b'} \widehat{b \otimes b'},$$

$$1 = \sum_{\alpha} \sum_{b \in B(\alpha)} \widehat{b \otimes b^\vee},$$

$$\Delta \left( \sum_{b, b'} a_{b, b'} \widehat{b \otimes b'} \right) = \sum_{b, b'} \sum_{d, d'} a_{b, d, d', b'} \widehat{b \otimes b' \otimes d \otimes d'}, \quad \epsilon(\widehat{b \otimes b'}) = \widehat{b \otimes b'}(b_0 \otimes b_0)$$

where  $B(0) = \{b_0\}$ .

**Definition** Let us denote by  $\tilde{e}_i, \tilde{f}_i, \text{wt}_i$  the following elements of  $\mathbb{B}^\vee$ :

$$\tilde{f}_i = \sum_{\alpha \in \Phi_+} \sum_{b \in B(\alpha)} \widehat{\tilde{f}_i b \otimes b^\vee} = \sum_{\alpha \in \Phi_+} \sum_{b \in B(\alpha)} b \widehat{(\tilde{e}_i b)^\vee},$$

$$\tilde{e}_i = \sum_{\alpha \in \Phi_+} \sum_{b \in B(\alpha)} \widehat{\tilde{e}_i b \otimes b^\vee} = \sum_{\alpha \in \Phi_+} \sum_{b \in B(\alpha)} b \widehat{(\tilde{f}_i b)^\vee},$$

$$\text{wt}_i = \sum_{\alpha \in \Phi_+} \sum_{b \in B(\alpha)} \text{wt}_i(b) \widehat{b \otimes b^\vee}.$$

**Remark** Then these satisfy the relations

$$[\tilde{e}_i, \text{wt}_j] = 2\delta_{i,j}\tilde{e}_i, [\tilde{f}_i, \text{wt}_j] = -2\delta_{i,j}\tilde{f}_i, [\text{wt}_i, \text{wt}_j] = 0.$$

We can see that  $\tilde{e}_i(b \otimes b') = \delta_{\tilde{e}_i b, b'^\vee}$ ,  $\tilde{f}_i(b \otimes b') = \delta_{\tilde{f}_i b, b'^\vee}$ ,  $\text{wt}_i(b \otimes b') = \text{wt}_i(b) b \otimes b'$ . By a slight abuse of notation, if we let  $\mathcal{F}(\tilde{e}_i, \tilde{f}_i \mid i \in I) = \mathcal{F}(\tilde{e}_i, \tilde{f}_i)$  be the free monoid on these generators, then for  $x \in \mathcal{F}(\tilde{e}_i, \tilde{f}_i)$  we have  $x(b \otimes b') = \delta_{xb, b'^\vee}$ . Hence these can be thought of as the Kashiwara operators on a crystal. Indeed, each  $\mathbb{B}(\alpha)$  is a  $\mathbb{B}$ -comodule and hence a  $\mathbb{B}^*$ -module, where

$$\widehat{d \otimes d'} \cdot b = \sum_c \widehat{d \otimes d'}(b \otimes c^\vee) c = \delta_{d, b} d'$$

for  $b \in B(\alpha)$ . In particular, the action of Kashiwara operators agree with our usual notions.

**Definition** Consider the full subcategory of  $\mathbb{B}^\vee$ -modules  $M$  such that, for each  $m \in M$ ,  $\widehat{b \otimes b'} \cdot m \neq 0$  for only finitely many  $b \otimes b'$  and  $(\sum_{b,b'} a_{b,b'} \widehat{b \otimes b'}) \cdot m = \sum_{b,b'} a_{b,b'} \widehat{b \otimes b'} \cdot m$  for each formal sum  $\sum_{b,b'} a_{b,b'} \widehat{b \otimes b'} \in \mathbb{B}^\vee$ , which we shall denote  $\mathbb{B}^\vee \text{Mod}^f$ .

**Remark** We may give each  $M$  in  $\mathbb{B}^\vee \text{Mod}^f$  the structure of a  $\mathbb{B}$ -comodule via

$$\Delta_M(m) = \sum_{b,b'} b \otimes b' \otimes (\widehat{b \otimes b'} \cdot m).$$

This gives a functor  $\mathbb{B}^\vee \text{Mod}^f \rightarrow \mathbb{B} \text{CoMod}$ , since module homomorphisms induce comodule homomorphism. Likewise, given a  $\mathbb{B}$ -comodule  $C$  we may endow this with a  $\mathbb{B}^\vee$ -module structure via

$$\mathbb{B}^\vee \otimes C \xrightarrow{\Delta_C} \mathbb{B}^\vee \otimes \mathbb{B} \otimes C \xrightarrow{\text{ev} \otimes \text{id}} C.$$

It is clear that this gives a quasi-inverse functor  $\mathbb{B} \text{CoMod} \rightarrow \mathbb{B}^\vee \text{Mod}^f$ , hence these categories are equivalent. We can also reformulate our classification of based comodules as crystals in this setting.

**Definition** We say that a pair  $(M, X)$  is a based  $\mathbb{B}^\vee$ -module if  $M \in \mathbb{B}^\vee \text{Mod}^f$  and  $X$  is a basis of  $M$  such that  $X = \sqcup_\alpha \sqcup_b X_b^\alpha$  where  $X_b^\alpha = X \cap M_b^\alpha$  and  $M_b^\alpha = \widehat{b \otimes b^\vee} \cdot M$ , and  $\widehat{b \otimes b^\vee}$  gives a bijection  $X_{b'}^\alpha \rightarrow X_b^\alpha$ . Morphisms between based modules are morphisms of modules that preserve the bases.

We then have the following restatement of Theorem 6.8.

**Theorem 6.11.** *Based  $\mathbb{B}^\vee$  modules are equivalent, as a category, to the category of crystals.*

**Remark** Not all  $\mathbb{B}^\vee$ -modules are in  $\mathbb{B}^\vee \text{Mod}^f$ . For example,  $\text{wt}_i \in \mathbb{B}^\vee$  has  $\widehat{b \otimes b'} \cdot \text{wt}_i = \text{wt}_i(b') \widehat{b \otimes b'} \neq 0$  for infinitely many  $b \otimes b'$ .

**Remark** For the remainder of this section we follow a similar construction to that of the bialgebra  $\dot{U}$  in [12, p. 183]. Consider the non-unital subalgebra  $\dot{U}_0 = \bigoplus_{\alpha \in \Phi_+} \mathbb{B}(\alpha) \otimes \mathbb{B}(-\alpha)$  of  $\mathbb{B}^\vee$ . In fact  $\dot{U}_0$  is a bi-ideal, and hence a  $\mathbb{B}^\vee$ -bimodule. Although this algebra does not have a unit, it has a family  $(1_\alpha)_{\alpha \in \Phi_+}$  where  $1_\alpha = \sum_{b \in B(\alpha)} \widehat{b \otimes b^\vee}$  that acts like a unit. For any  $u \in \dot{U}_0$  there is a finite subset  $\Xi \subset \Phi_+$  such that  $(\sum_{\alpha \in \Xi} 1_\alpha) \cdot u = u$ . Lusztig calls such a collection  $(1_\alpha)_{\alpha \in \Phi_+}$  a *generalised unit* in [13].

**Definition** We say that a  $\dot{U}_0$ -module  $M$  is *unital* if, for any  $m \in M$  there are finitely many  $\alpha \in \Phi_+$  such that  $1_\alpha \cdot m \neq 0$ , and  $\sum_{\alpha \in \Phi_+} 1_\alpha \cdot m = m$ .

**Remark** For a unital  $A$ -module  $M$ , we have  $M = \bigoplus_{\alpha \in \Phi} M^\alpha$  where  $M^\alpha = 1_\alpha \cdot M$ . Thus  $M$  becomes a  $\mathbb{B}^\vee$ -module via  $x \cdot m := (x \cdot 1_\alpha) \cdot m$  for  $x \in \mathbb{B}^\vee$ ,  $m \in M^\alpha$ , where the product  $x \cdot 1_\alpha$  is taken in  $\mathbb{B}^\vee$ . For each  $\widehat{b \otimes b'} \in \dot{U}_0$ , with  $b \in B(\beta)$ ,  $b' \in B(-\beta)$ ,  $\widehat{b \otimes b'} \cdot M^\alpha = 0$  whenever  $\beta \neq \alpha$ . Thus each  $M^\alpha$ , and hence  $M$ , becomes a  $\mathbb{B}^\vee$ -module in  $\mathbb{B}^\vee \text{Mod}^f$ . It is easy to see that, likewise,



$\mathbb{B}^\vee$ -modules in  $\mathbb{B}^\vee \text{Mod}^f$  form unital  $\dot{U}_0$ -modules. This gives an equivalence of categories, hence we obtain the following result.

**Proposition 6.12.** *The category of based unital  $\dot{U}_0$ -modules is equivalent to the category of crystals, where based  $\dot{U}_0$ -modules are defined analogously to based  $\mathbb{B}^\vee$ -modules.*

**Remark** Note that  $\dot{U}_0$  does not form a subcoalgebra, since

$$\Delta(\widehat{b_0 \otimes b_0^\vee}) = \sum_{b, b', d, d'} \delta_{b, d, b_0} \delta_{d', b', b_0} b \otimes b' \otimes d \otimes d'$$

contains all terms where  $b = b'^\vee$  is of highest weight and  $d = d'^\vee$  is of lowest weight. However, we do have a collection of maps

$$\begin{aligned} \Delta_{\beta, \beta'}^\alpha : \mathbb{Q}B(\alpha) \otimes \mathbb{Q}B(-\alpha) &\rightarrow \mathbb{Q}B(\beta) \otimes \mathbb{Q}B(-\beta) \otimes \mathbb{Q}B(\beta') \otimes \mathbb{Q}B(-\beta'), \\ \widehat{b \otimes b'} &\mapsto \sum_{d, d' \in B(\beta)} \sum_{d'', d''' \in B(\beta')} \delta_{d, d'', b} \delta_{d', d''', b'^\vee} d \otimes d' \otimes d'' \otimes d'''. \end{aligned}$$

These maps can be considered as a single map  $\dot{U}_0 \rightarrow \mathbb{B}^\vee \otimes \mathbb{B}^\vee$  that agrees with the restriction of the comultiplication.

**Proposition 6.13.** *With  $\dot{U}_0$  as above and  $\langle -, - \rangle$  the restriction of the dual pairing, we have*

$$\begin{aligned} \langle (b \otimes b') \cdot (c \otimes c'), \widehat{d \otimes d'} \rangle &= \langle (b \otimes b') \otimes (c \otimes c'), \Delta_{\beta, \beta'}^\alpha(\widehat{d \otimes d'}) \rangle, \\ \langle b \otimes b', 1_\alpha \rangle &= \delta_{\alpha, \beta} \epsilon(b \otimes b'), \end{aligned}$$

where  $b \otimes b' \in B(\beta) \otimes B(-\beta)$ ,  $c \otimes c' \in B(\beta') \otimes B(-\beta')$  and  $c \otimes c' \in B(\alpha) \otimes B(-\alpha)$ , and  $\langle -, - \rangle$  defines a bilinear form  $(\mathbb{B} \otimes \mathbb{B}) \times (\dot{U}_0 \otimes \dot{U}_0) \rightarrow \mathbb{Z}$ ,  $\langle x \otimes y, u \otimes v \rangle = \langle x, u \rangle \langle y, v \rangle$ . Furthermore, this defines a non-degenerate pairing  $\mathbb{B} \times \dot{U}_0 \rightarrow \mathbb{Z}$ .

**Definition** Let  $\tilde{e}_{\alpha, i} = \tilde{e}_i \cdot 1_\alpha = \sum_{b \in B(\alpha)} \tilde{e}_i \widehat{b \otimes b^\vee}$ ,  $\tilde{f}_{\alpha, i} = \tilde{f}_i \cdot 1_\alpha = \sum_{b \in B(\alpha)} \tilde{f}_i \widehat{b \otimes b^\vee}$  in  $\dot{U}_0$ , so  $\tilde{e}_i = \sum_\alpha \tilde{e}_{\alpha, i}$  and  $\tilde{f}_i = \sum_\alpha \tilde{f}_{\alpha, i}$  in  $\mathbb{B}^*$ . Then  $\tilde{e}_{\alpha, i}$  and  $\tilde{f}_{\alpha, i}$  act as  $\tilde{e}_i$  and  $\tilde{f}_i$  on  $\mathbb{B}(\alpha)$ , and by zero on  $\mathbb{B}(\beta)$  for  $\beta \neq \alpha$ .

**Proposition 6.14.**  *$\dot{U}_0$  is generated as an algebra by  $\{\tilde{e}_{\alpha, i}, \tilde{f}_{\alpha, i} \mid i \in I, \alpha \in \Phi_+\}$  along with the unit elements  $\{1_\alpha \mid \alpha \in \Phi_+\}$ .*

*Proof.* Fix  $\alpha \in \Phi_+$ . For  $i \in I$ ,  $1_\alpha - \tilde{f}_{\alpha, i} \tilde{e}_{\alpha, i}$  is the sum of  $\widehat{b \otimes b^\vee}$  such that  $\tilde{e}_i b = 0$ . So, for any ordering of  $I$ ,  $\prod_{i \in I} (1_\alpha - \tilde{f}_{\alpha, i} \tilde{e}_{\alpha, i})$  is the sum of  $b \otimes b^\vee$  where  $\tilde{e}_i b = 0$  for all  $i \in I$ . That is,  $b_\alpha \otimes b_\alpha^\vee = \prod_{i \in I} (1_\alpha - \tilde{f}_{\alpha, i} \tilde{e}_{\alpha, i})$ . The result then follows from the fact that if  $b = \tilde{f}_{i_1} \tilde{f}_{i_2} \dots \tilde{f}_{i_n} b_\alpha$  and  $b' = \tilde{f}_{j_1} \tilde{f}_{j_2} \dots \tilde{f}_{j_m} b_\alpha$  then  $\widehat{b \otimes b'^\vee} = \tilde{f}_{\alpha, i_1} \tilde{f}_{\alpha, i_2} \dots \tilde{f}_{\alpha, i_n} (b_\alpha \otimes b_\alpha^\vee) \tilde{e}_{\alpha, j_1} \tilde{e}_{\alpha, j_2} \dots \tilde{e}_{\alpha, j_m} = \tilde{f}_{\alpha, i_1} \dots \tilde{f}_{\alpha, i_n} \left( \prod_{i \in I} (1_\alpha - \tilde{f}_{\alpha, i} \tilde{e}_{\alpha, i}) \right) \tilde{e}_{\alpha, j_1} \dots \tilde{e}_{\alpha, j_m}$   $\square$

**6.4. Relation to Global Bases.** Kashiwara shows in [9] that crystal bases  $B(\alpha)$  of representations  $V(\alpha)$  induce global bases of the vector spaces  $V(\alpha)$ . Using these bases, we see that  $\mathcal{B}$  gives rise to a global base of  $A_q(\mathfrak{g})$ .

**Remark** Recall from Proposition 6.6 that, in the case of  $\mathfrak{sl}_2$ , the bialgebra  $\mathbb{B}$  is a quotient of the free algebra  $\mathbb{Z}\langle a, b, c, d \rangle$  by the relations

$$cb = bc = db = dc = ba = ca = 0, \quad da = 1,$$

viewed as a bialgebra via the comultiplication

$$\Delta(a) = a \otimes a + b \otimes c, \Delta(b) = a \otimes b + b \otimes d,$$

$$\Delta(c) = c \otimes a + d \otimes c, \Delta(d) = c \otimes b + d \otimes d.$$

In this case, the quantum coordinate ring  $A_q(\mathfrak{sl}_2)$  can be realised as a quotient of the free algebra  $K\langle a, b, c, d \rangle$  by the relations

$$db = qbd, dc = qcd, ba = qab, ca = qac$$

$$da = qcb + 1, cb = bc = qad - q1,$$

again viewed as a bialgebra with the analogous comultiplication as above. Kashiwara shows in [10] that  $\{a^i c^j d^k \mid i, j, k \geq 0\} \cup \{a^i b^j d^k \mid i, j, k \geq 0, j \neq 0\}$  is the global basis of  $A_q(\mathfrak{sl}_2)$  corresponding to the crystal base  $\mathcal{B}$ . We also saw in Proposition 6.6 that  $\mathcal{B}$  can be rewritten as  $\{a^i c^j d^k \mid i, j, k \geq 0\} \cup \{a^i b^j d^k \mid i, j, k \geq 0, j \neq 0\}$ , since  $a^i c^j d^k = x^{i+j} y^k \otimes (x^i y^{j+k})^\vee$  and  $a^i b^j d^k = x^i y^{j+k} \otimes (x^{i+j} y^k)^\vee$ . It is then apparent that the multiplication in  $\mathbb{B}$  on basis elements is the result of multiplication of the corresponding global basis elements and taking only the  $q^0$  coefficient when written in terms of the global basis (that is, evaluating at  $q = 0$ ). A similar result can be formulated for the comultiplication.

It is a goal of future work by the author to investigate whether this phenomenon is exclusive to  $\mathfrak{sl}_2$ . In [13], Lusztig does something similar, using the multiplication of global basis elements (or *canonical basis* elements) of a modified version of  $U_q(\mathfrak{g})$ , denoted  $\dot{U}$ , to define a bialgebra. He refers to his construction as a quantum group at  $v = \infty$ , but it could equally be considered a quantum group at  $q = v^{-1} = 0$ . Since  $\dot{U}$  is dual to  $A_q(\mathfrak{g})$ , this bialgebra should be dual to  $\mathbb{B}$  and should give some way of describing the (co)multiplication of  $\mathbb{B}$  in terms of the (co)multiplication of global basis elements of  $A_q(\mathfrak{g})$ .

## REFERENCES

- [1] Alain Bruguières, Steve Lack & Alexis Virelizier, Hopf Monads on Monoidal Categories, *Advances in Mathematics*, Vol. 227, 2 (2011), 745-800.
- [2] Francis Borceaux, *Handbook of Categorical Algebra 2*, Cambridge University Press, Encyclopedia of Mathematics and its Applications, 51 (1994).
- [3] C. Geiß, B. Leclerc, J. Schröer, Cluster structures on quantum coordinate rings, *Selecta Mathematica*, Vol. 19, 2 (2013), 337-397.
- [4] Jens Carsten Jantzen, *Lectures on Quantum Groups*, American Mathematical Society, Graduate Studies in Mathematics, 6 (1995).

- [5] André Henriques & Joel Kamnitzer, Crystals and Coboundary Categories, *Duke Mathematical Journal*, Vol. 132 (2006), 191-216.
- [6] Masaki Kashiwara Crystal bases of modified quantized enveloping algebra, *Duke Mathematical Journal*, Vol. 73, 2 (1994), 383-413.
- [7] Masaki Kashiwara, Crystallizing the q-analogue of universal enveloping algebras, *Communications in Mathematical Physics*, Vol. 133 (1990) 249-260.
- [8] Masaki Kashiwara, On Crystal Bases, [www.kurims.kyoto-u.ac.jp/kenkyubu/kashiwara/oncrystal.pdf](http://www.kurims.kyoto-u.ac.jp/kenkyubu/kashiwara/oncrystal.pdf), 1995.
- [9] Masaki Kashiwara, On crystal bases of the q-analogue of universal enveloping algebras, *Duke Mathematical Journal*, Vol. 63 (1991), 465-516.
- [10] Masaki Kashiwara, Global crystal bases of quantum groups, *Duke Mathematical Journal*, Vol. 69, 2 (1993), 455-485.
- [11] Christian Kassel, *Quantum Groups*, Springer-Verlag, Graduate Texts in Mathematics, 155 (1995).
- [12] George Lusztig, *Introduction to Quantum Groups*, Birkhäuser, Progress in Mathematics, 110 (1993).
- [13] George Lusztig, Quantum groups at  $v = \infty$ , *Functional Analysis on the Eve of the 21st Century Volume 1*, Birkhäuser, Progress in Mathematics, 131 (1995), 199-221.
- [14] Paddy McCrudden, Opmonoidal monads, *Theory and Applications of Categories*, Vol. 10, 19 (2002), 469-485.
- [15] I. Moerdijk, Monads on Tensor Categories, *Journal of Pure and Applied Algebra*, Vol. 168, 2-3 (2002), 189-208.
- [16] Alain Bruguières & Alexis Virelizier, Hopf Monads, *Advances in Mathematics*, Vol. 215, 2 (2007), 679-733.
- [17] Zoran Škoda, Monadic Functor, <http://ncatlab.org/nlab/show/monadic+functor>, accessed 07 February 2014.